Fock–Goncharov dual cluster varieties and Gross–Siebert mirrors

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Plan of the talk

- Cluster varieties
  - Fock Goncharov dual cluster varieties: $(\mathcal{A}, \mathcal{X})$
- Gross–Siebert mirror symmetry for (log) Calabi–Yau varieties

Main result:
The mirror of a log Calabi–Yau compactification of the $\mathcal{X}$ cluster variety is a degeneration of the $\mathcal{A}$ cluster variety and vice-versa.
Argüz–Bousseau, arXiv:2206.10584

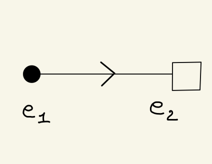
Cluster varieties from seeds

- Fomin–Zelevinsky, Fock–Goncharov,...

Definition

A *seed* is the data of:

- a lattice $N \cong \mathbb{Z}^n$, with a basis $\{e_i\}_{i \in \bar{I}}$
- a skew-symmetric form $\langle \cdot, \cdot \rangle : N \times N \to \mathbb{Z}$



$\#$ arrows $= \langle e_i, e_j \rangle$

Mutation of a seed at the vertex e_k:

$$e'_i = \begin{cases} e_i + \max(\langle e_i, e_k \rangle, 0) e_k & \text{if } i \neq k \\ -e_k & \text{if } i = k \end{cases}$$

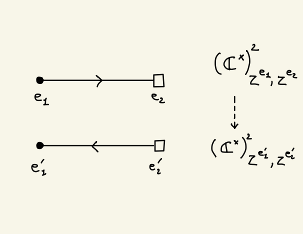
Cluster varieties from seeds

The $\mathcal{X}$ and $\mathcal{A}$ cluster varieties are obtained by gluing tori:

- The $\mathcal{X}$ cluster variety is a union of tori: $\bigcup \operatorname{Spec} \mathbb{C}[N]$ glued via birational transformations $z^n \mapsto z^n(1+z^{e_k})^{-(v_k,n)}$
- The $\mathcal{A}$ cluster variety is a union of tori: $\bigcup \operatorname{Spec} \mathbb{C}[M]$ glued via birational transformations $z^m \mapsto z^m(1+z^{v_k})^{-(e_k,m)}$, where $v_k = \langle e_k, - \rangle \in M$.

Fix $I \subset \bar{I}$, and denote

$$N_{uf} = \bigoplus_{i \in I} \mathbb{Z} e_i$$



The Fock–Goncharov conjecture

- Fock–Goncharov: $\mathcal{A}$ and $\mathcal{X}$ are dual in the sense that the algebra of regular functions on $\mathcal{A}$ admits a canonical basis indexed by integral tropical points of $\mathcal{X}$ (and vice-versa).
- Gross–Hacking–Keel [1]: The Fock–Goncharov conjecture cannot hold without necessary positivity assumptions (roughly put, unless $\mathcal{X}$ is affine)
- Gross–Hacking–Keel–Kontsevich [2]: proof of the Fock–Goncharov conjecture, under necessary positivity assumptions

[1] Gross–Hacking–Keel: Birational geometry of cluster algebras, **Algebraic Geometry**, (2015)

[2] Gross-Hacking-Keel-Kontsevich: Canonical bases for cluster algebras, **Journal of the American Mathematical Society**, (2018).

Proof of the Fock–Goncharov conjecture without positivity assumptions

- Consider the $\mathcal{A}$ cluster variety with principal coefficients $\mathcal{A}_{prin} \to \operatorname{Spec} \mathbb{C}[N_{uf}]$
  - $\mathcal{A}_{prin}$ is a family containing the $\mathcal{A}$ cluster variety.
  - It is obtained from the seed for the $\mathcal{A}$ cluster variety, by adding one frozen vertex for every unfrozen vertex
- There is a partial compactification $\bar{\mathcal{A}}_{prin} \to \operatorname{Spec} \mathbb{C}[N_{uf}^+]$ with special fiber the torus $\operatorname{Spec} \mathbb{C}[M]$
  - Let $\widehat{\bar{\mathcal{A}}_{prin}} \to \operatorname{Spf} \mathbb{C}[\![N_{uf}^+]\!]$, formal completion along special fiber.

Theorem (GHKK[1])

The $\mathbb{C}[\![N_{uf}^+]\!]$-algebra of regular functions on $\widehat{\bar{\mathcal{A}}_{prin}}$ admits a canonical basis indexed by integral tropical points of $\mathcal{X}$.

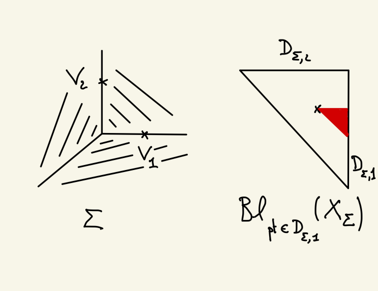
- The proof uses a "cluster scattering diagram".

[1] Gross-Hacking-Keel-Kontsevich: Canonical bases for cluster algebras, **Journal of the American Mathematical Society**, (2018)

Cluster varieties as blow-ups of toric varieties

From the data of a seed one can construct a toric variety as follows:

- $v_i := \langle e_i, - \rangle \in M$ (assume v_i is primitive for all $i \in I$ – otherwise, work with orbifolds).
- Σ : a fan in $M_{\mathbb{R}}$ of a smooth projective toric variety X_Σ containing the rays $\mathbb{R}_{\geq 0} v_i$ for all $i \in \overline{I}$.
- $D_{\Sigma,i}$: the toric divisor of X_Σ corresponding to the ray $\mathbb{R}_{\geq 0} v_i$.
- H_i : hypersurface of $D_{\Sigma,i}$ defined by the equation $1 + z^{e_i} = 0$, for all $i \in I$ (unfrozen).



Log Calabi–Yau compactifications of cluster varieties

Theorem (Gross–Hacking–Keel [2])

Let (X, D) be the log Calabi–Yau pair where X is the blow-up of X_Σ along the hypersurfaces $H_i \subset D_{\Sigma,i}$, and D is the strict transform of D_Σ. The $\mathcal{X}$ cluster variety is isomorphic (up to codimension two) to $X \setminus D$.

- (X, D) is called a log Calabi-Yau compactification of the $\mathcal{X}$ cluster variety. An analogous construction (exchanging e_i and v_i), defines a log CY compactification of the $\mathcal{A}$ cluster variety.
- Gross–Siebert[3]: Mirror symmetry for more general log Calabi–Yau pairs (X, D) in any dimension.

[1] Gross–Hacking–Keel: Birational geometry of cluster algebras, **Algebraic Geometry**, (2015)

[2] Gross-Siebert: The canonical wall structure and intrinsic mirror symmetry, **Inventiones mathematicae**, (2022)

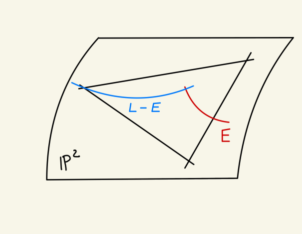
[3] Argüz-Gross: The higher dimensional tropical vertex, **Geometry & Topology**, (2020).

- Gross–Siebert mirror symmetry: Associated to a (maximal) log Calabi–Yau pair (X, D) is a mirror family

$$X^\vee \longrightarrow \mathrm{Spf}\mathbb{C}[\![\mathrm{NE}(X)]\!]$$

where $\mathrm{NE}(X)$ is the monoid of integral points in the cone of effective curve classes in X, constructed using a **canonical wall structure**[1] which is a combinatorial gadget encoding tropical analogues of "$\mathbb{A}^1$ curves" in the "tropicalization of (X, D)".

- $\mathbb{A}^1$ curves: stable rational maps in (X, D) with image touching D at a single point)

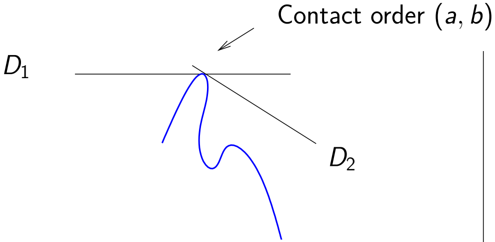


The blow up of the projective space $\mathbb{CP}^2$ at one point on D.

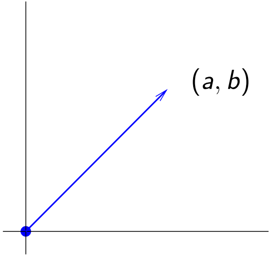
- E: exceptional curve
- $L-E$: the strict transform of the line passing through the point we blow up and the opposite vertex.

The tropicalization of (X, D)

- The *tropicalization of* (X, D), $\Sigma_{(X,D)}$ is a topological space referred to as a *conical affine pseudo-manifold*[1].
  - The integral points parametrize possible tangency conditions at a point of intersection of the curve with D.
  - If D is smooth, we just have $\Sigma_{(X,D)}(\mathbb{Z}) = \mathbb{N}$.
  - If D has two components meeting transversally at a nodal point, then $\Sigma_{(X,D)}(\mathbb{Z}) = \mathbb{N}^2$.



$D = D_1 \amalg D_2$



[1] Gross–Hacking–Keel–Siebert: Theta functions on varieties with effective anti-canonical class, **Memoirs of the AMS**, (2016)

The canonical wall structure

Definition (Gross-Siebert)

The canonical wall structure $\mathfrak{D}_{(X,D)}$ associated to (X, D) is a union of pairs $(\mathfrak{d}, f_{\mathfrak{d}})$, where

- $\mathfrak{d}$: codimension one cells in the tropicalization of (X, D)
- $f_{\mathfrak{d}}$: wall crossing functions given by

$$f_{\mathfrak{d}} = \exp(k_\tau N_{\tau,\beta} t^\beta z^{-v}),$$

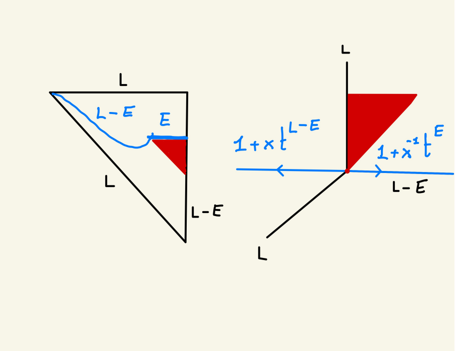
  where τ is a tropical type of an $\mathbb{A}^1$ curve, $k_\tau \in \mathbb{N}$ is a multiplicity, $\beta \in H_2(X, \mathbb{Z})$, $N_{\tau,\beta}$ is the number of $\mathbb{A}^1$ curves of type τ and class β, and v is the direction of the leg of the tropical curve corresponding to the point of intersection with the boundary.

- Heuristically, $\mathbb{A}^1$ curves are analogues of holomorphic discs with boundaries on SYZ fibers, the class β keeps track of the area of such discs, while v is the homology class of the boundary.

The canonical wall structure for the blow up of $\mathbb{P}^2$ at a non toric point

Example

X: the blow-up of $\mathbb{P}^2$ at a single non-toric point, D strict transform of the toric boundary divisor in $\mathbb{P}^2$. The canonical wall structure for (X, D):



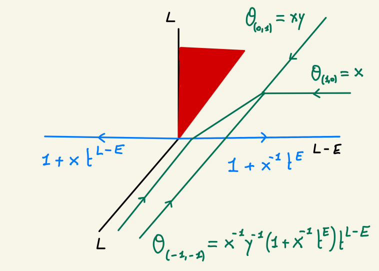
Theta functions from broken lines

- Algebra of functions on the mirror of (X, D): $\mathbb{C}[\![NE(X)]\!]$-algebra with (topological) basis of theta functions $(\vartheta_m)_{m \in \Sigma_{(X,D)}(\mathbb{Z})}$

Example

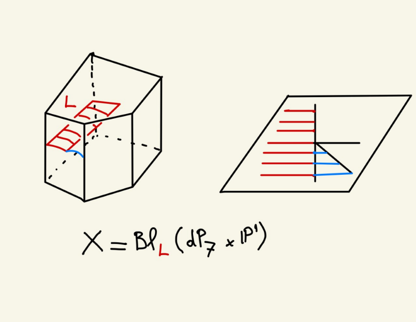
X: the blow-up of $\mathbb{P}^2$ at a single non-toric point, D strict transform of the toric boundary divisor in $\mathbb{P}^2$. The mirror to (X, D):

$$\operatorname{Spec} \mathbb{C}[NE(X)][\vartheta_{(1,0)}, \vartheta_{(0,1)}, \vartheta_{(-1,-1)}] / \left(\vartheta_{(1,0)}\vartheta_{(0,1)}\vartheta_{(-1,-1)} = t^{[L]} + t^{L-E}\vartheta_{(1,0)}\right)$$

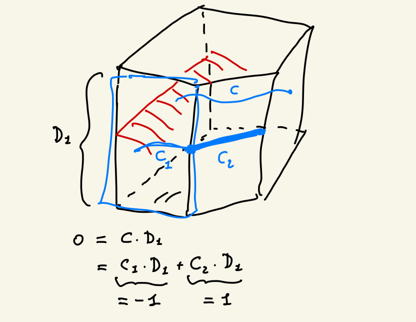


Negative orders in higher dimensions!

- X: Blow up of $dP_7 \times \mathbb{P}^1$ along a line, illustrated in red.
- Curves illustrated in blue have negative contact orders.



Negative orders in higher dimensions!



- Counts of curves with negative contact orders along D give "punctured log Gromov–Witten invariants" [1]

[1] Abramovich-Chen-Gross-Siebert: Punctured logarithmic maps, arXiv:2009.07720.

Challenge: practically computing the mirror generally!

To write explicit compute the mirror to (X, D), one needs to compute all possible $\mathbb{A}^1$-curves on (X, D).

If (X, D) has a toric model, there is a combinatorial algorithm to compute all $\mathbb{A}^1$-curves in (X, D) in any dimension. [1]

[1] Argüz–Gross: The higher dimensional tropical vertex, **Geometry & Topology**, (2020).

Computing $\mathbb{A}^1$ curves: wall structures

To compute $\mathbb{A}^1$ curves on (X, D) "combinatorially", we construct a wall structure in $\mathbb{R}^n$:

Definition (Kontsevich–Soibelman / Gross–Siebert)

Fix $M \cong \mathbb{Z}^n$, $N = \mathrm{Hom}(M, \mathbb{Z})$. A **wall structure** (scattering diagram) in $\mathbb{R}^n$ is a union of pairs $(\mathfrak{d}, f_{\mathfrak{d}})$ where

- $\mathfrak{d}$: codimension one subset of $M_{\mathbb{R}} = \mathbb{R}^n$
- $f_{\mathfrak{d}} \in G$, where G is the Lie group associated to the Lie algebra of zero-divergence vector fields on a family of tori,

$$\mathfrak{g} = \bigoplus_{m \in M, n \in N, (m,n)=0} \mathbb{C}[[t]] z^m \partial_n$$

with the Lie bracket given by $[z^m \partial_n, z^{m'} \partial_{n'}] = z^{m+m'} \partial_{(m',n)n' - (m,n')n}$

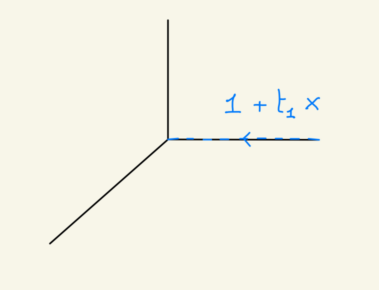
Example: wall structures

Assume, X: blow-up of a toric X_Σ along hypersurfaces $H_i \subset D_i$ in the toric boundary D_Σ, and D: strict transform of D_Σ. We define an **initial wall structure** $\mathfrak{D}_{(X_\Sigma,H),\mathrm{in}}$ given by a union of pairs $(\mathfrak{d}_i, f_{\mathfrak{d}_i})$, where

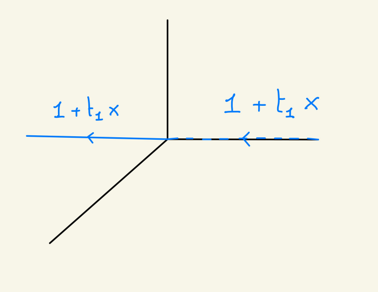
- $\mathfrak{d}_i$: the "**deformation space**"[1] of the tropicalization of H_i,
- $f_{\mathfrak{d}_i} := 1 + t_i z^{m_i}$, where m_i is the direction of the ray corresponding to D_i and t_i is a formal variable.

Example

Let $X_\Sigma = \mathbb{P}^2$, and H one non-toric point. Then we have $\mathfrak{D}_{(X_\Sigma,H),\mathrm{in}}$:



- **Crossing walls**: To a wall $(\mathfrak{d}, f_{\mathfrak{d}})$ with a normal vector n, associate a wall crossing automorphism $z^{m_i} \mapsto f^{\langle m_i, n\rangle} z^{m_i}$.
- A scattering diagram is called **consistent** if for any joint (codim 2 locus where walls intersect), the composition of all wall-crossing transformations on all adjacent walls are identity.
- The initial wall structure $\mathfrak{D}_{(X_\Sigma, H),\mathrm{in}}$ is not consistent!
    - But we can complete it to a consistent one, $\mathfrak{D}_{(X_\Sigma, H)}$ algorithmically!



The higher dimensional tropical vertex

- The wall structure $\mathfrak{D}_{(X_\Sigma, H)}$ constructed combinatorially, determines the canonical wall structure defined using enumerative geometry.
  - From t_i variables to curve class variables: $f_\mathfrak{d}$ is necessarily a power-series with monomials of the form $\prod_i (t_i z^{m_i})^{a_i}$. Set $\mathbf{A} = \{a_i\}$ and

$$\bar{\beta}_{\mathbf{A}} := \psi(-\sum_i a_i m_i) + \sum_i \psi(a_i m_i)\,,$$

    where $\psi \colon M_\mathbb{R} \to N_1(X_\Sigma) \otimes \mathbb{R}$ with kink along a codimension one cone ρ being the class of the corresponding one-dimensional stratum $D_\rho \subset X$.
  - Define

$$\beta_{\mathbf{A}} := \bar{\beta}_{\mathbf{A}} - \sum_i a_i E_i$$

- Replace $\prod_i (t_i z^{m_i})^{a_i}$ by $t^{\beta_{\mathbf{A}}} z^{\sum_i a_i m_i}$

The higher dimensional tropical vertex

Theorem (Argüz–Gross)

Let (X, D) be a log Calabi-Yau pair with toric model (X_Σ, D_Σ), and let $\dim X = n$*. In the associated consistent scattering diagram, let $(\mathfrak{d}_{out}, f_{\mathfrak{d}_{\mathrm{out}}})$ be a wall produced by the algorithm. Then,*

$$\log f_{\mathfrak{d}_{out}} = \sum_{\tau} \sum_{\beta} k_\tau N_{\tau,\beta}(X, D) t^\beta z^{-u_\tau}$$

- *τ is a tropical curve with one leg having direction vector u_τ, and a $(n-2)$-dimensional deformation space which forms $\mathfrak{d}_{out}$,*
- *k_τ is a constant number associated to τ,*
- *$\beta \in H_2(X, \mathbb{Z})$,*
- *$N_{\tau,\beta}(X, D)$ is the count of $\mathbb{A}^1$ curves on (X, D) of class β and tropicalization τ.*

Idea of proof

- Construct a degeneration to the normal cone $(\widetilde{X}, \widetilde{D})$ to (X, D). The central fiber is a union of
  - The toric variety X_Σ
  - Blow-ups of $\mathbb{P}^1$ bundles over D_Σ
- Degeneration formula: counts of $\mathbb{A}^1$ curves in (X, D) can be computed from counts of rational curves in (X_Σ, D_Σ) with relative tangency conditions and the multiple cover formula.

Example

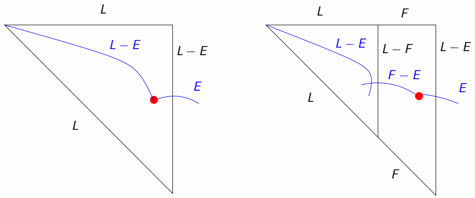


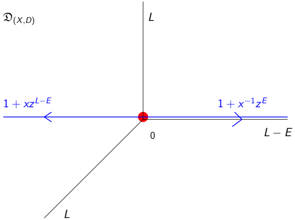
Figure: A degeneration of the blow-up of $\mathbb{P}^2$ at one point to the union of $\mathbb{P}^2$ and the blow-up of a Hirzebruch surface

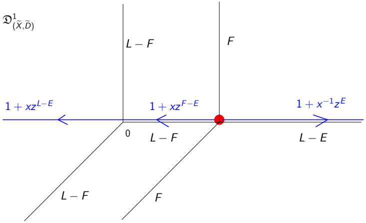
A combinatorial construction of the canonical wall structure

- The tropicalization $\widetilde{X}$ has a natural projection map to $\mathbb{R}_{\geq 0}$.
  - $\mathfrak{D}^1_{(\widetilde{X},\widetilde{D})}$: walls of $\mathfrak{D}_{(\widetilde{X},\widetilde{D})}$ on the height one slice (this is an affine manifold with singularities away from the origin).
  - $T_0\mathfrak{D}^1$: The localization around the origin of $\mathfrak{D}^1_{(\widetilde{X},\widetilde{D})}$.

$$\boxed{T_0\mathfrak{D}^1_{(\widetilde{X},\widetilde{D})} \cong \mathfrak{D}_{(X_\Sigma,H)}} \text{ and } \boxed{\mathfrak{D}^{1,\text{asymptptotic}}_{(\widetilde{X},\widetilde{D})} \cong \mathfrak{D}_{(X,D)}}$$

  - Doing monodromy analysis we conclude $\mathfrak{D}_{(X,D)}$ can be determined from $\mathfrak{D}_{(X_\Sigma,H)}$.





Extension of the Gross–Siebert mirror family

- (X, D): HDTV log Calabi-Yau pair
- $\pi : (X, D) \to (X_\Sigma, D_\Sigma)$, blow-up of hypersurfaces $H_i \subset D_\Sigma$.
  - Gross–Siebert mirror family: $X^\vee \to \mathrm{Spf}\, \mathbb{C}[\![NE(X)]\!]$
- We show this family extends!
  - Exceptional divisors: $\mathbb{P}^1$-bundles $\mathcal{E}_i \to H_i$.
  - $\mathbb{P}^1$-fibers E_i.
  - $N_1(X) = \pi^* N_1(X_\Sigma) \oplus \bigoplus_i \mathbb{Z} E_i$ group of curve classes modulo numerical equivalence
  - Define the monoid $Q \subset N_1(X)$,

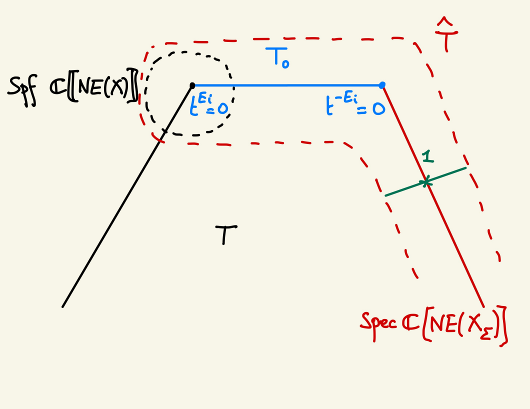
$$Q := \pi^* NE(X_\Sigma) \oplus \bigoplus_i (-\mathbb{N}) E_i$$
$$= \{\pi^* \bar{\beta} - \sum_i a_i E_i \mid \bar{\beta} \in NE(X_\Sigma), a_i \geq 0\}$$

- Define the toric variety T by gluing the affine toric varieties $\mathrm{Spec}\, \mathbb{C}[NE(X)]$ and $\mathrm{Spec}\, \mathbb{C}[Q]$ along their common intersection

$$\mathrm{Spec}\, \mathbb{C}[\pi^* NE(X_\Sigma) \oplus \bigoplus_i \mathbb{Z} E_i]$$

- Natural loci in T:
  - $T_0 := \prod_i \mathbb{P}^1$ with coordinates t^{E_i} and t^{-E_i}
  - $\operatorname{Spec} \mathbb{C}[NE(X_\Sigma)]$
- Define $\widehat{T}$ as the formal completion of T along

$$T_0 \cup \operatorname{Spec} \mathbb{C}[NE(X_\Sigma)] .$$



Extension of the mirror family

Theorem (Argüz-Bousseau[1])

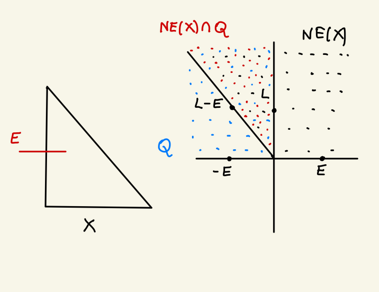
The mirror family $X^\vee \to \mathrm{Spf}\, \mathbb{C}[\![NE(X)]\!]$ *canonically extends to a family* $\widetilde{X^\vee} \to \widehat{T}$. *In other words, all curve classes appearing in the equations of the mirror of* X *are elements of* $NE(X) \cap Q$.
Moreover, the restriction of this extended mirror family to $\mathrm{Spec}\, \mathbb{C}[NE(X_\Sigma)]$ *is the toric mirror family* $X_\Sigma \to \mathrm{Spec}\, \mathbb{C}[NE(X_\Sigma)]$ *of the toric variety* X_Σ.

- Mirror symmetry explanation: this deformation from the mirror of X to the mirror of X_Σ is mirror to the blow-up $X \to X_\Sigma$.

[1] Argüz-Bousseau: Fock-Goncharov dual cluster varieties and Gross-Siebert mirrors, arXiv:2206.10584

Extension of the mirror family: example

- (X, D) obtained from toric $\mathbb{P}^2$ by one non-toric blow-up.
  - L: pullback of the class of a line in $\mathbb{P}^2$
  - E: class of the exceptional curve



- Recall the equation of the mirror: $\vartheta_1 \vartheta_2 \vartheta_3 = t^L + t^{L-E} \vartheta_1$. Indeed, only curve classes in $NE(X) \cap Q$: L and $L - E$ appear.

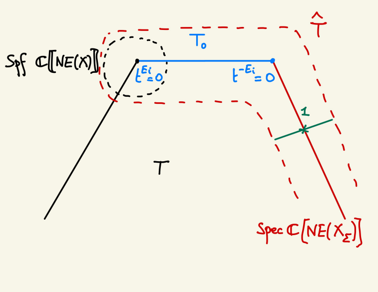
Application: mirror symmetry for cluster varieties

- (X, D): a log Calabi-Yau compactification of the $\mathcal{X}$ cluster variety.
  - (X, D) is an example of HDTV log Calabi-Yau pair.
  - Gross-Siebert mirror construction: mirror family $X^\vee \to \mathrm{Spf}\, \mathbb{C}[\![NE(X)]\!]$.
  - How does it compare with the Fock-Goncharov dual $\mathcal{A}$ cluster variety?
- Keel-Yu[1]: For their non-archimedean construction, if $\mathcal{X}$ is affine, then the mirror family extends over $\mathrm{Spec}\, \mathbb{C}[NE(X)]$ and one recovers $\mathcal{A}$ as the fiber over 1 in the big torus orbit of the toric variety $\mathrm{Spec}\, \mathbb{C}[NE(X)]$
  - concretely, set all curve classes β in the mirror equation to 0.
- What if $\mathcal{X}$ is not affine?
  - In general, no extension over $\mathrm{Spec}\, \mathbb{C}[NE(X)]$.
  - Setting all curve classes to 0 produces divergent series.

[1] Keel-Yu: The Frobenius structure theorem for affine log Calabi-Yau varieties containing a torus, arXiv:1908.09861 (2019).

Application: mirror symmetry for cluster varieties

- We set all curve classes coming from the toric variety X_Σ to zero (this makes sense in the extended family $\widetilde{X^\vee} \to \hat{T}$).
  - We obtain a family over $\mathrm{Spf}\, \mathbb{C}[\![N_{uf}^+]\!]$.



Theorem (Argüz-Bousseau[1])

The restriction to $\mathrm{Spf}\, \mathbb{C}[\![N_{uf}^+]\!]$ of the extended mirror family to the log Calabi-Yau compactification (X, D) of the $\mathcal{X}$-cluster variety is the formal completion $\widehat{\bar{\mathcal{A}}_{prin}} \to \mathrm{Spf}\, \mathbb{C}[\![N_{uf}^+]\!]$ of the $\mathcal{A}$ cluster variety with principical coefficients.

[1] Argüz-Bousseau: Fock-Goncharov dual cluster varieties and Gross-Siebert mirrors, arXiv:2206.10584

Thank You!