An invitation to motivic sheaves II

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A. Motivic sheaves. One hopes that for any scheme S and an appropriate commutative ring A (say $A = \mathbb{Z}, \mathbb{Z}/n, \mathbb{Q}, \bar{\mathbb{Q}}$) there exists certain abelian A-category $M(S, A)$ of (mixed) motivic (perverse) A-sheaves over S together with corresponding derived category $DM(S,A)$.
This categories should resemble very much the categories of mixed ℓ-adic sheaves;

- there should be inner $\otimes$ and Hom
- there should be Tate sheaves $A_{\mathcal{M}}(i)$; the Tate twist $M \longmapsto M(i) = M \otimes A_{\mathcal{M}}(i)$ is automorphism of $M(S,A)$
- for any morphism $S_1 \to S_2$ of finite type there should be the corresponding functors $f_*, f^*, f_!, f^!$ between $DM(S_i)$
- for $A \supset \mathbb{Q}$ there should be canonical weight filtration $W_\cdot$ on the objects of $M(S,A)$ such that any morphism is strictly compatible with $W_\cdot$ and any $Gr^W_\cdot(M)$ is semisimple.

All this things should behave the same way as in mixed ℓ-adic situation. One should also have realisation functors r on $M(S)$ and $DM(S)$ with values in mixed ℓ-adic sheaves, or Hodge sheaves (if S

Beilinson 1985

Ext groups = motivic cohomology

$$D(\mathcal{MM}(S)) \supseteq \mathcal{MM}(S) \supseteq \mathcal{M}(S)$$

"projective" objects

$$D(\mathrm{Coh}(S)) \supseteq \mathrm{Coh}(S) \supseteq \mathrm{Vect}(S)$$

(S affine)

Ext groups = coherent cohomology

$$H^q_{\mathcal{M}}(S, \mathbb{Q}(p)) := \mathrm{Ext}^q_{\mathcal{MM}(S)_{\mathbb{Q}}}(M(S), M(S)_{\mathbb{Q}}(p))$$

This should be a "universal" cohomology theory.

Realization functors

$$\mathcal{MM}(S) \longrightarrow \mathrm{Shv}(S)$$

$\mathrm{Shv}(S_{\text{ét}})$

$\mathrm{Shv}(S(\mathbb{C}))$

induce maps on Ext^*.

$$\Rightarrow \quad H^*_{\mathcal{M}}(S, \mathbb{Q}(*)) \longrightarrow H^*_{\text{ét}}(S, \mathbb{Q}(*))$$

$$H^*_{\mathcal{M}}(S, \mathbb{Q}(*)) \longrightarrow H^*_{\text{sing}}(S(\mathbb{C}), \mathbb{Q}(*))$$

$$\begin{tikzcd} CH^n(S)_{\mathbb{Q}} \arrow[r] \arrow[dr] & H^{2n}_{\acute{e}t}(S, \mathbb{Q}_\ell(n)) \\ & H^{2n}_{sing}(S, \mathbb{Q}) \end{tikzcd}$$

Beilinson's candidate:

$$H^q_{\mu}(S, \mathbb{Q}(p)) \simeq Gr^p_{\gamma} K_{2p-q}(S)_{\mathbb{Q}}$$

Grothendieck-Riemann-Roch:

$$\begin{tikzcd}
K_0(S)_{\mathbb{Q}} \arrow[r, phantom, "\simeq"] \arrow[d, phantom, sloped, "\cong"] & \bigoplus_n \mathrm{Gr}_\gamma^n K_0(S)_{\mathbb{Q}} \arrow[d, phantom, sloped, "\cong"] \\
CH^*(S)_{\mathbb{Q}} \arrow[r, phantom, "\simeq"] & \bigoplus_n CH^n(S)_{\mathbb{Q}}
\end{tikzcd}$$

$$K_0(X) = K_0(\mathrm{Coh}(X))$$

generated by coherent sheaves $\mathcal{F} \in \mathrm{Coh}(X)$

modulo relations

$$[\mathcal{F}] = [\mathcal{F}'] + [\mathcal{F}'']$$

for all exact sequences

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0.$$

$K_i(X)$ = higher algebraic K-theory

(Quillen)

Bloch (1986) + Levine (1994):

higher Chow groups $CH^q(X, p)$

$$CH^*(X, p)_{\mathbb{Q}} \xrightarrow{\sim} Gr_\gamma^* K_p(X)_{\mathbb{Q}}$$

$$H^q_{\mathcal{M}}(X, \mathbb{Q}(p)) \simeq CH^p(X, 2p-q)_{\mathbb{Q}}$$

$$H^{BM,\mathcal{M}}_q(X, \mathbb{Q}(p)) \simeq CH_q(X, 2p+q)_{\mathbb{Q}}$$

In "classical" (co)homology,
we have localization long exact sequences,
Mayer-Vietoris, cohomological descent, ...

In our "algebraic" homology theories,
we only have:

$$CH_*(Z) \to CH_*(X) \to CH_*(X \setminus Z) \to 0$$

$$K_0(Z) \to K_0(X) \to K_0(X \setminus Z) \to 0$$

The "higher" theories correct this:

$$\cdots \xrightarrow{\partial} CH_q(Z,p) \to CH_q(X,p) \longrightarrow CH_q(X \setminus Z, p)$$
$$\xrightarrow{\partial} CH_q(Z, p-1) \longrightarrow \cdots$$

$$\cdots \xrightarrow{\partial} K_p(Z) \longrightarrow K_p(X) \longrightarrow K_p(X \setminus Z) \xrightarrow{\partial} \cdots$$

$K_*(X)$ and $CH_q(X,*)$ are the homotopy groups of ∞-groupoids $\leftrightarrow$ "homotopy types"

$K(X)$ and $z_q(X)$.

At the level of ∞-groupoids, we have fibre sequences:

$$Z_q(Z) \longrightarrow Z_q(X) \longrightarrow Z_q(X \setminus Z)$$

$$K(Z) \longrightarrow K(X) \longrightarrow K(X \setminus Z)$$

$Z_q(X)$ = impose rational equivalence rel'n on algebraic cycles up to homotopy

$K(X)$ = impose $[\mathcal{F}] = [\mathcal{F}'] + [\mathcal{F}'']$ rel'ns on coherent sheaves up to homotopy

Let A be a set.

$R \subseteq A \times A$ equivalence relation

$\rightsquigarrow \quad R \mathrel{\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow}} A$

$$A/R = \operatorname{colim}\left(R \mathrel{\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow}} A\right) \cong \operatorname{coeq}\left(R \rightrightarrows A\right)$$

We can think of $R \rightrightarrows A$ as

presenting a groupoid G with

$$\mathrm{Ob}(G) = A$$

$$\mathrm{Mor}(G) = R$$

We have $\pi_0(G) = A/R$.

set of connected components

More generally, we can consider

simplicial diagrams:

$$\cdots \quad \substack{\rightarrow\\ \leftarrow\\ \rightarrow\\ \leftarrow\\ \rightarrow\\ \leftarrow\\ \rightarrow} \; A_2 \; \substack{\rightarrow\\ \leftarrow\\ \rightarrow\\ \leftarrow\\ \rightarrow} \; A_1 \; \substack{\rightarrow\\ \leftarrow\\ \rightarrow} \; A_0$$

which leads us to ∞-groupoids.

coequalizers $\rightsquigarrow$ "geometric realizations" of simplicial diagrams standard algebraic n-simplex

$$\Delta^n_k = \mathrm{Spec}\left(k[t_0, \dots, t_n] / \left(\textstyle\sum t_i - 1\right)\right) \quad \left(\simeq \mathbb{A}^n_k\right)$$

There are face and degeneracy maps between Δ^n_k's which lead to a simplicial diagram

$$\cdots \; \substack{\to\\\to\\\to\\\to} \; Z^*(X \times_k \Delta^2_k) \; \substack{\to\\\to\\\to} \; Z^*(X \times_k \Delta^1_k) \; \substack{\to\\\leftarrow\\\to} \; Z^*(X \times_k \Delta^0_k).$$

$Z^*(X)$
$= \{$alg. cycles on $X\}$

The resulting ∞-groupoid is $z^*(X)$.

Note: We need to look at subgroups of cycles which are in "good position", i.e. intersect properly with the faces.

Let us return to the question of constructing $DM(S)$.

Clausen-Scholze

Animation (a.k.a. derived categories in ∞-category theory)

Quillen, Lurie

$\mathcal{A}$ = a category of "finite projective" objects

There exists an ∞-category $\hat{\mathcal{A}}$ such that

- $\mathcal{A} = \{\text{compact projective objects in } \hat{\mathcal{A}}\} \subseteq \hat{\mathcal{A}}$
- The objects of $\hat{\mathcal{A}}$ are built out of filtered colimits ("unions of increasing towers") in $\mathcal{A}$ and geometric realizations (quotients of higher equiv. rel'ns) in $\mathcal{A}$

There exists a stable ∞-category $D(\mathcal{A})$ which is obtained by inverting the "suspension" functor in $\hat{\mathcal{A}}$.

Examples:

$\mathcal{A} = \{\text{finite sets}\}$

$$\rightsquigarrow \quad \hat{\mathcal{A}} = \{\infty\text{-groupoids}\} \quad (\text{homotopy types})$$

$$D(\mathcal{A}) = \{\text{spectra}\} \quad (\text{stable homotopy types})$$

$\mathcal{A} = \{\text{f.g. projective } R\text{-modules}\}$

$$\rightsquigarrow \quad \hat{\mathcal{A}} = D(R)_{\geq 0} \quad (= D(R)^{\leq 0})$$

$$D(\mathcal{A}) = D(R)$$

S scheme

Let $\mathcal{A} = \mathcal{M}(S) = \{\text{Chow motives over } S\}$

built out of "relative" correspondences

$\rightsquigarrow \quad DM(S) := D(\mathcal{A})$

Agrees with Voevodsky motives by work of Bondarko, F. Jin.

Voevodsky's construction:

Take $\mathcal{A} = Sm_S = \{\text{smooth } S\text{-schemes}\}$.

$$DM(S) := D(Sm_S) \Big/ \left\{ \begin{array}{ll} X \times \mathbb{A}^1 \to X & \forall X \in Sm_S \\ U \sqcup_{U \cap V} V \to X & \begin{tikzcd} U \cap V \arrow[r] \arrow[d] \arrow[dr, phantom, "\forall \mathrm{Nis}"] & V \arrow[d] \\ U \arrow[r] & X \end{tikzcd} \end{array} \right\} \left[(\mathbb{P}^1)^{\otimes -1} \right]$$

Verdier quotient

Define motivic cohomology using Ext's in DM.

Theorem (hard):

motivic cohomology $\simeq$ higher Chow groups

Theorem (Voevodsky, Ayoub, ...):

The construction $S \longmapsto DM(S)_{\mathbb{Q}}$

admits the six functor formalism.

Beilinson's conjectures on motivic sheaves

$$\Updownarrow$$

There is a motivic t-structure on $DM(S)$

(non-degenerate, compatible with perverse t-structures under realizations).

$$\Rightarrow \quad MM(S) := DM(S)^{\heartsuit}.$$

(hypothetical) abelian category of motivic sheaves over S

Theorem (Hanamura, Beilinson, Bondarko, ...):

Existence of the motivic t-structure

$\Updownarrow$

Grothendieck's standard conjectures

Hanamura

Beilinson

Bondarko

The conservativity conjecture:

The Betti realization functor

"geometric" motives

$$DM_{gm}(\mathbb{C})_{\mathbb{Q}} \longrightarrow D(\mathbb{Q})$$

is conservative.

Conservativity $\Longrightarrow$ Bloch conjecture

S sm proj surface $/\mathbb{C}$ $h^{2,0}(X) = 0$

$\mathrm{Alb} : CH_0(X) \longrightarrow \mathrm{Alb}(X)(k)$ is injective.