Moduli of relative stable maps to $\mathbb{P}^1$: cut-and-paste invariants

Siddarth Kannan

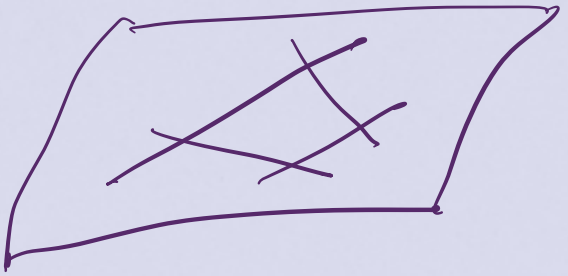
(Brown University)

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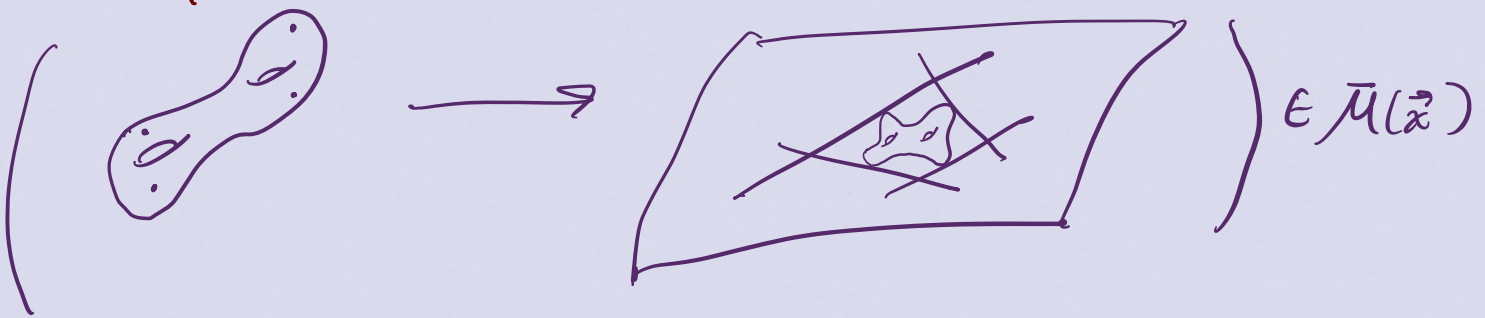
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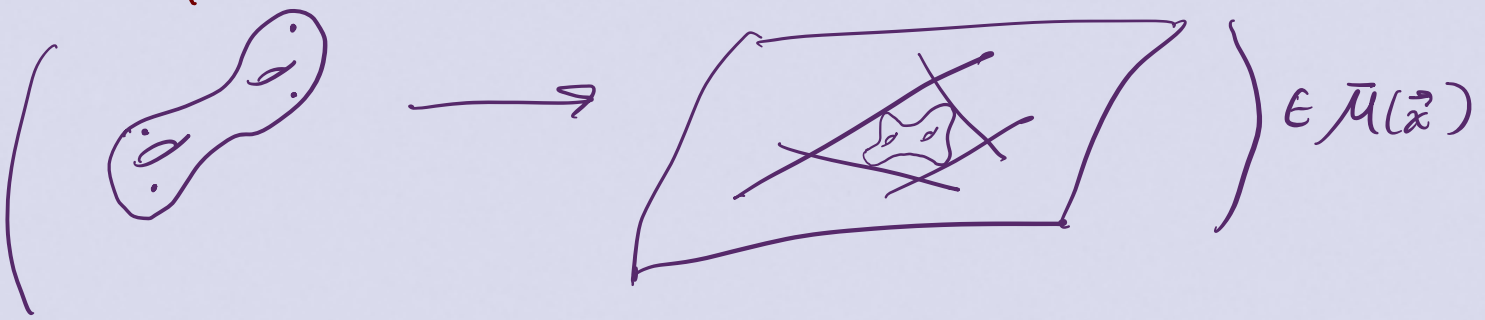
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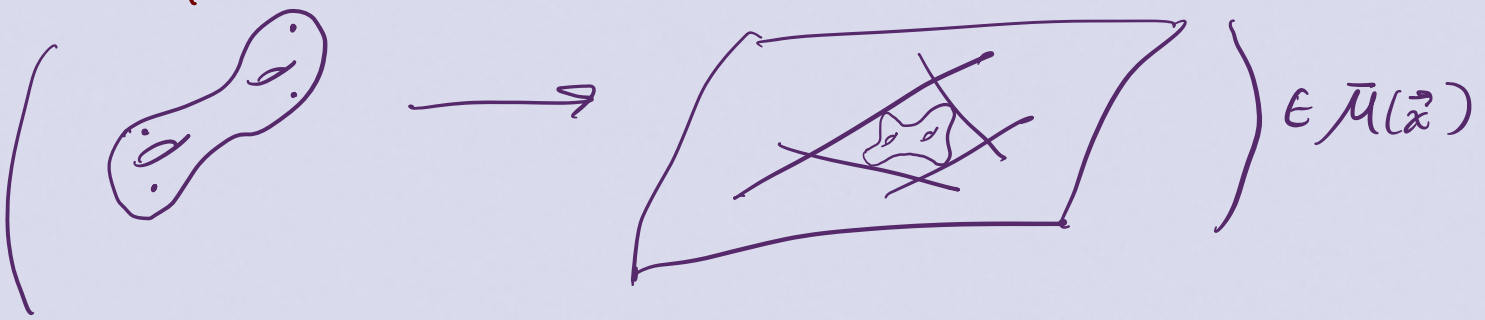


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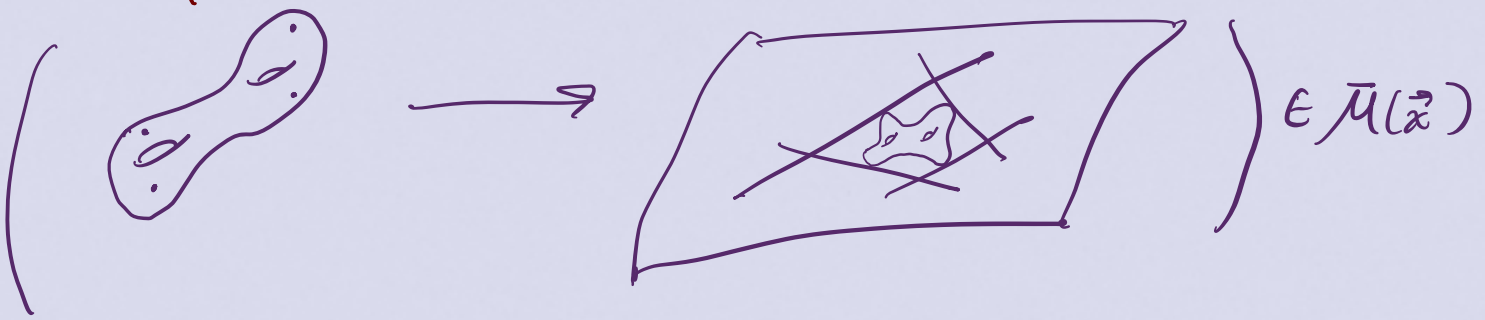


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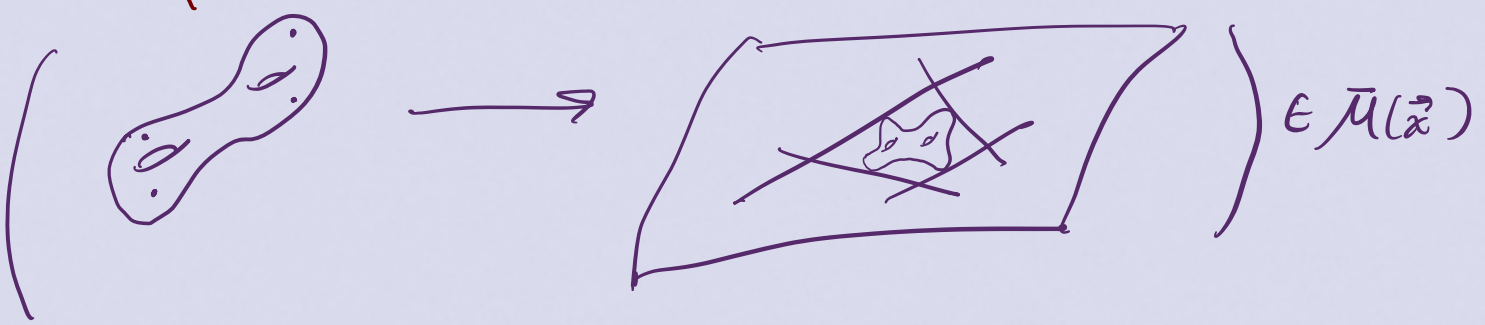
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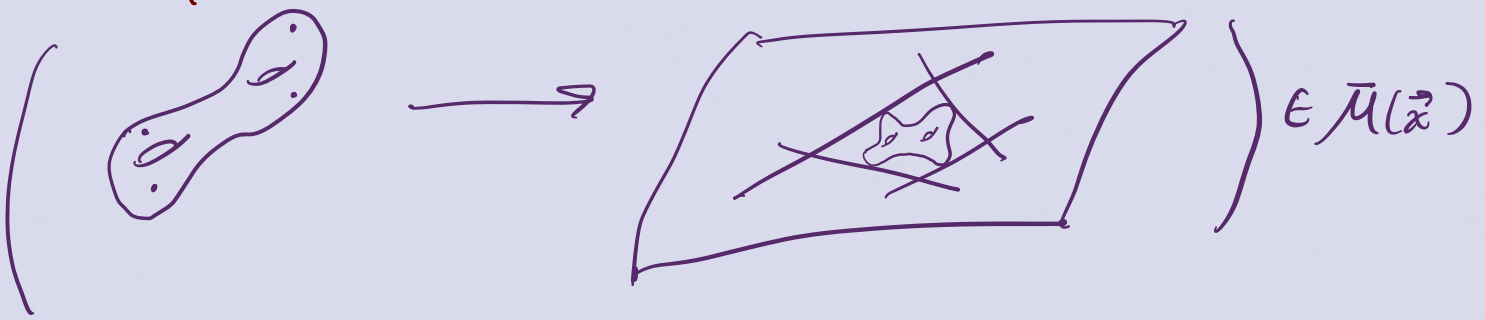
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Fix $\vec{x} = (x_1, \dots, x_n) \in (\mathbb{Z} \setminus \{0\})^n$ with $\sum x_i = 0$

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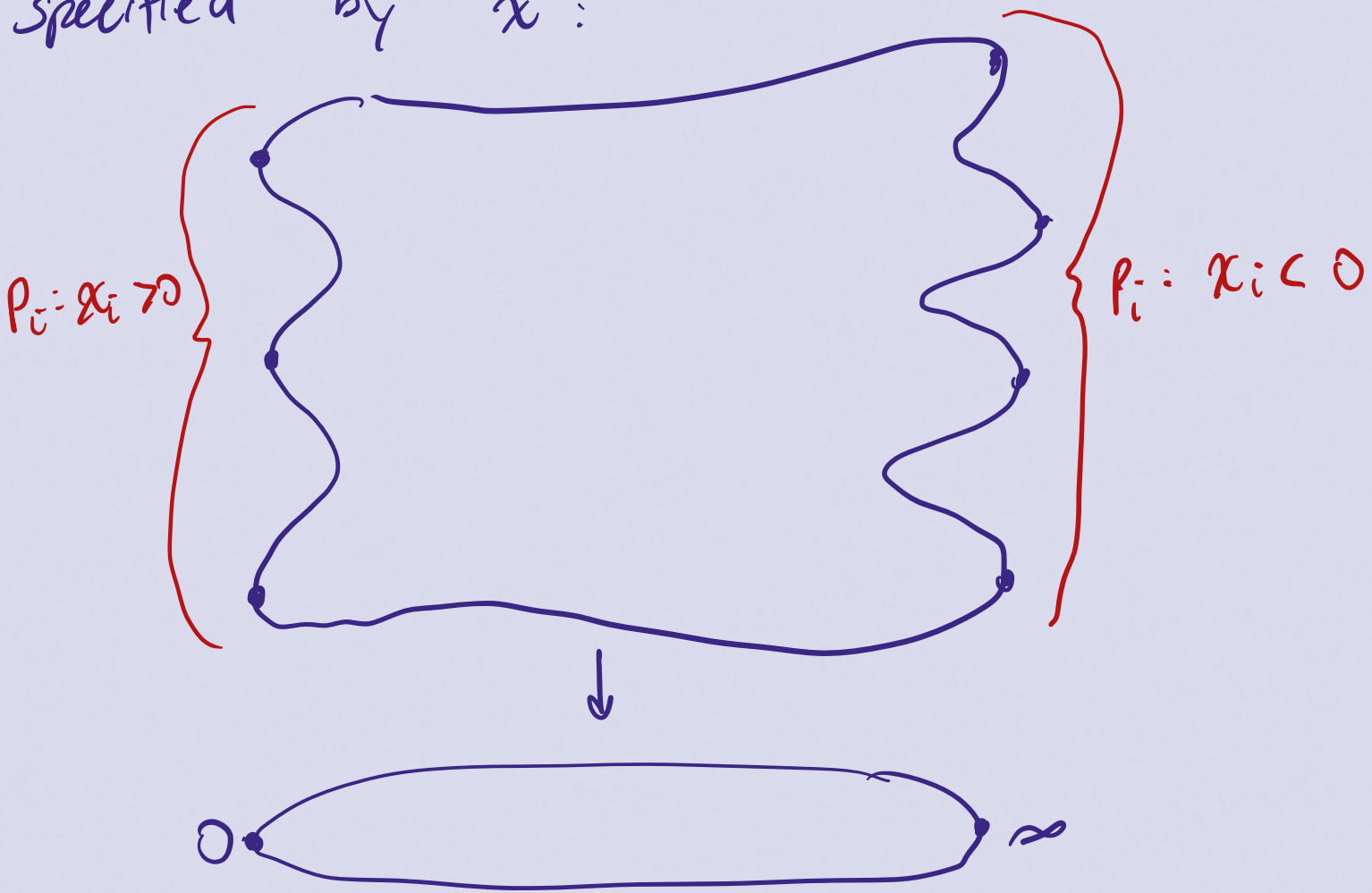
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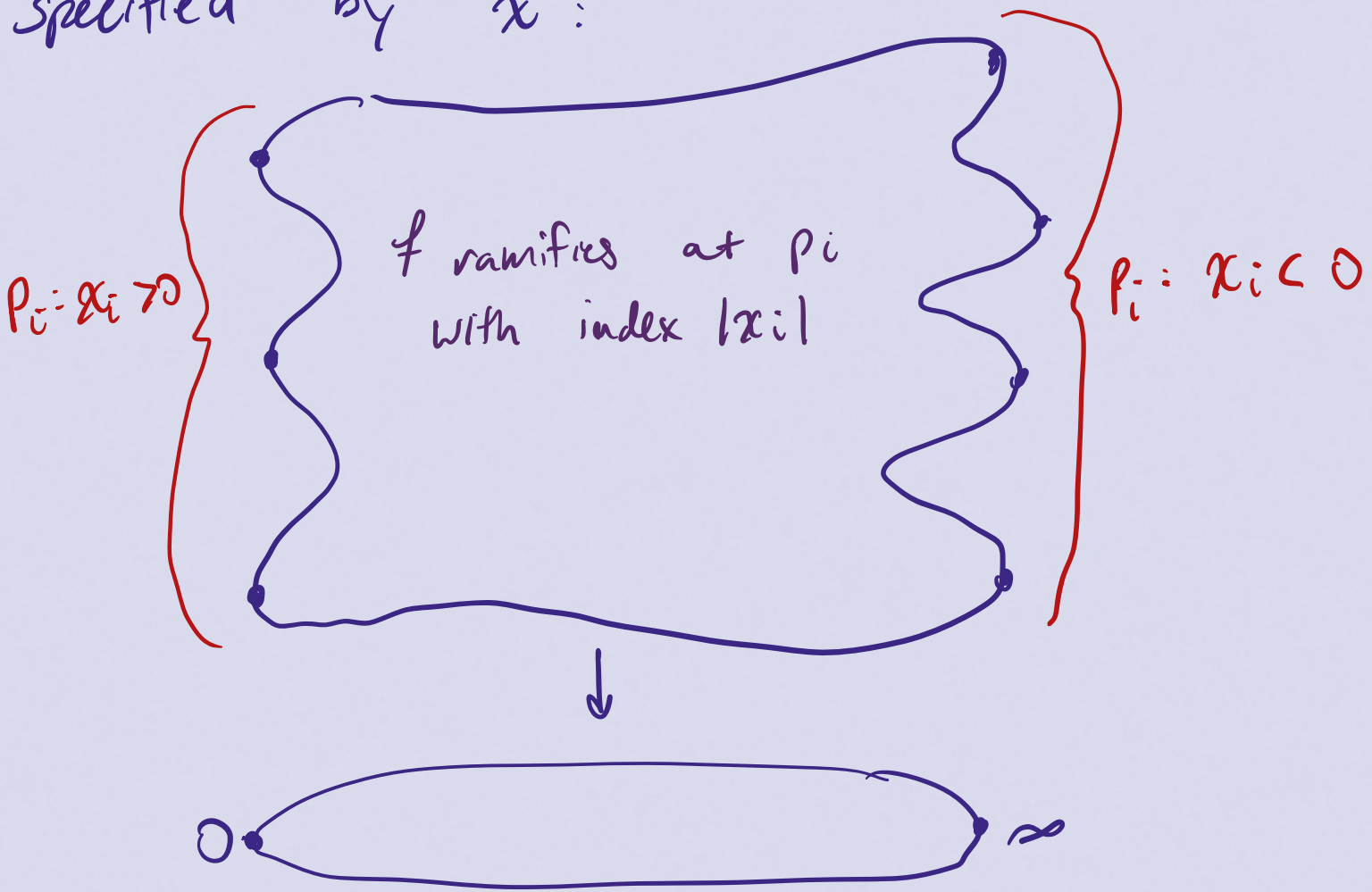
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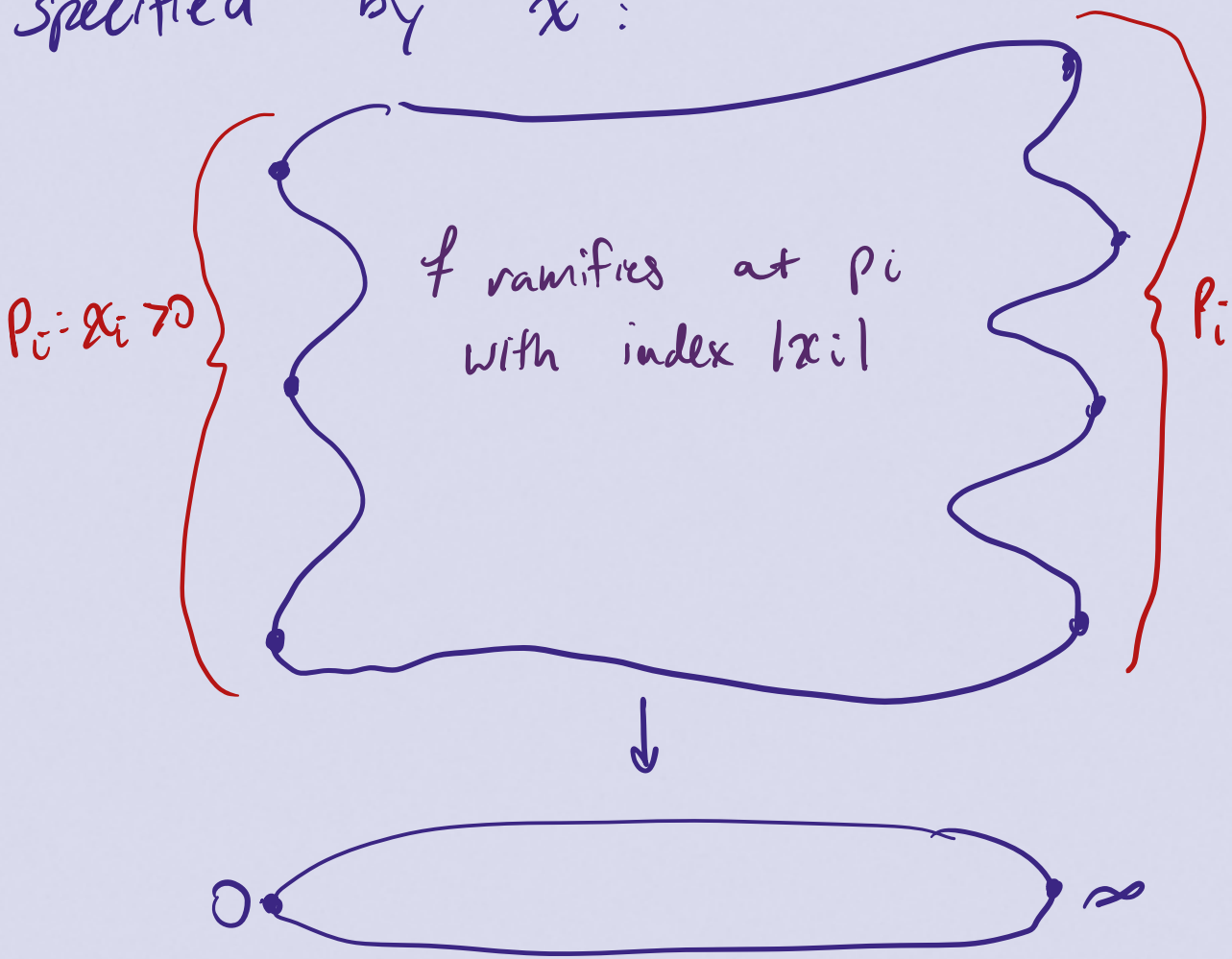
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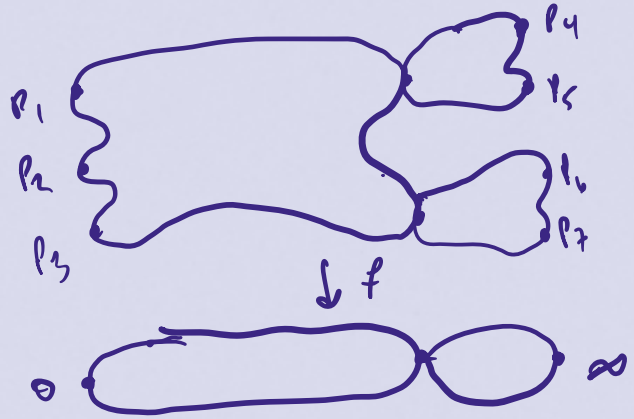
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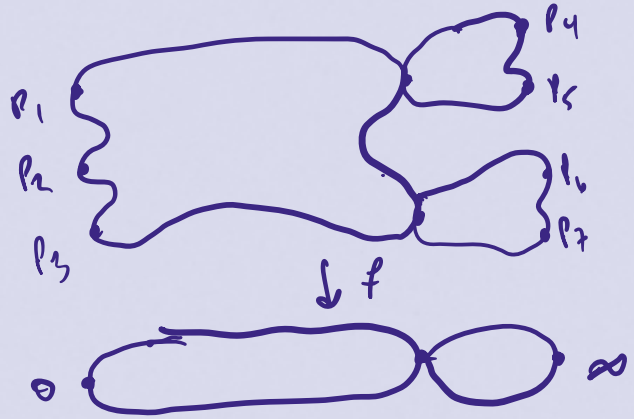


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In this way get a system $\bar{\mathcal{M}}(\vec{x})$ of moduli spaces, each birational to $\bar{\mathcal{M}}_{0,n}$.

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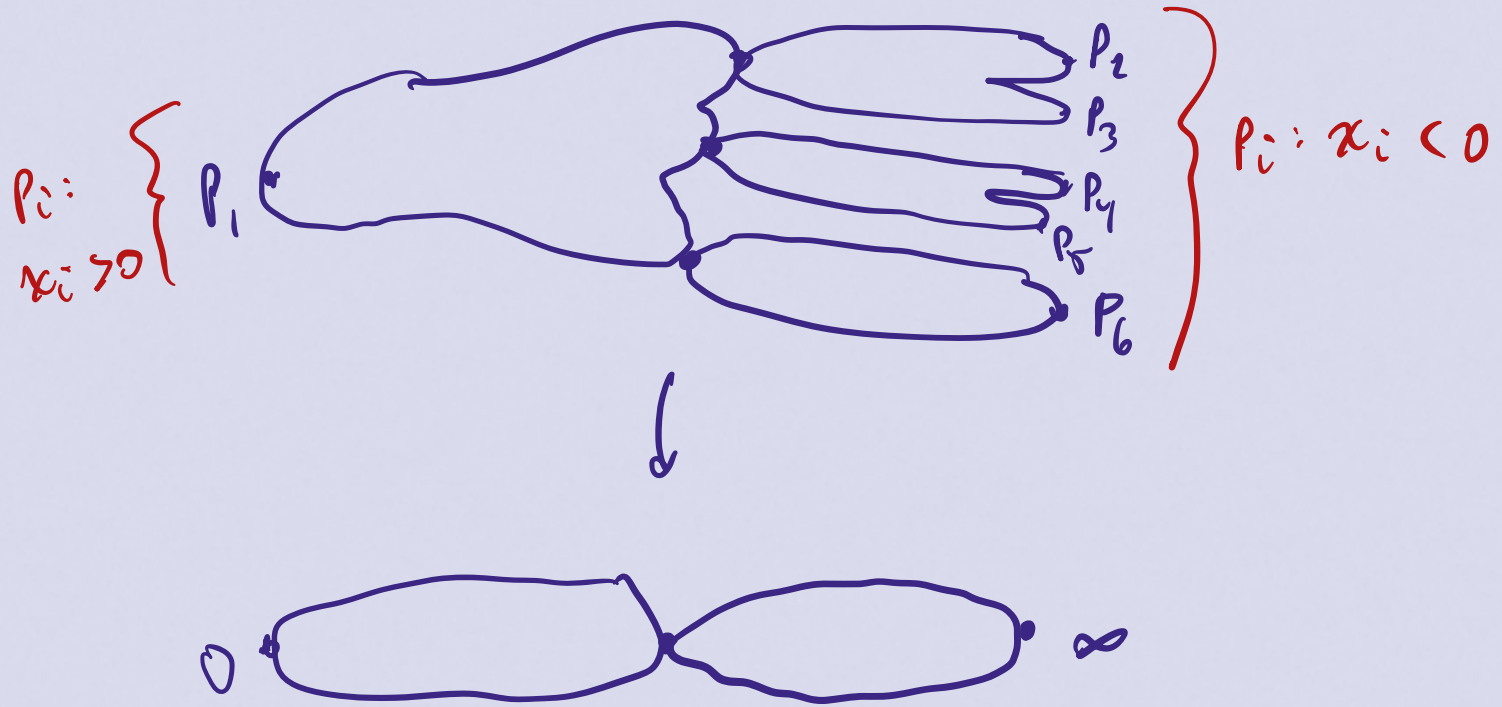
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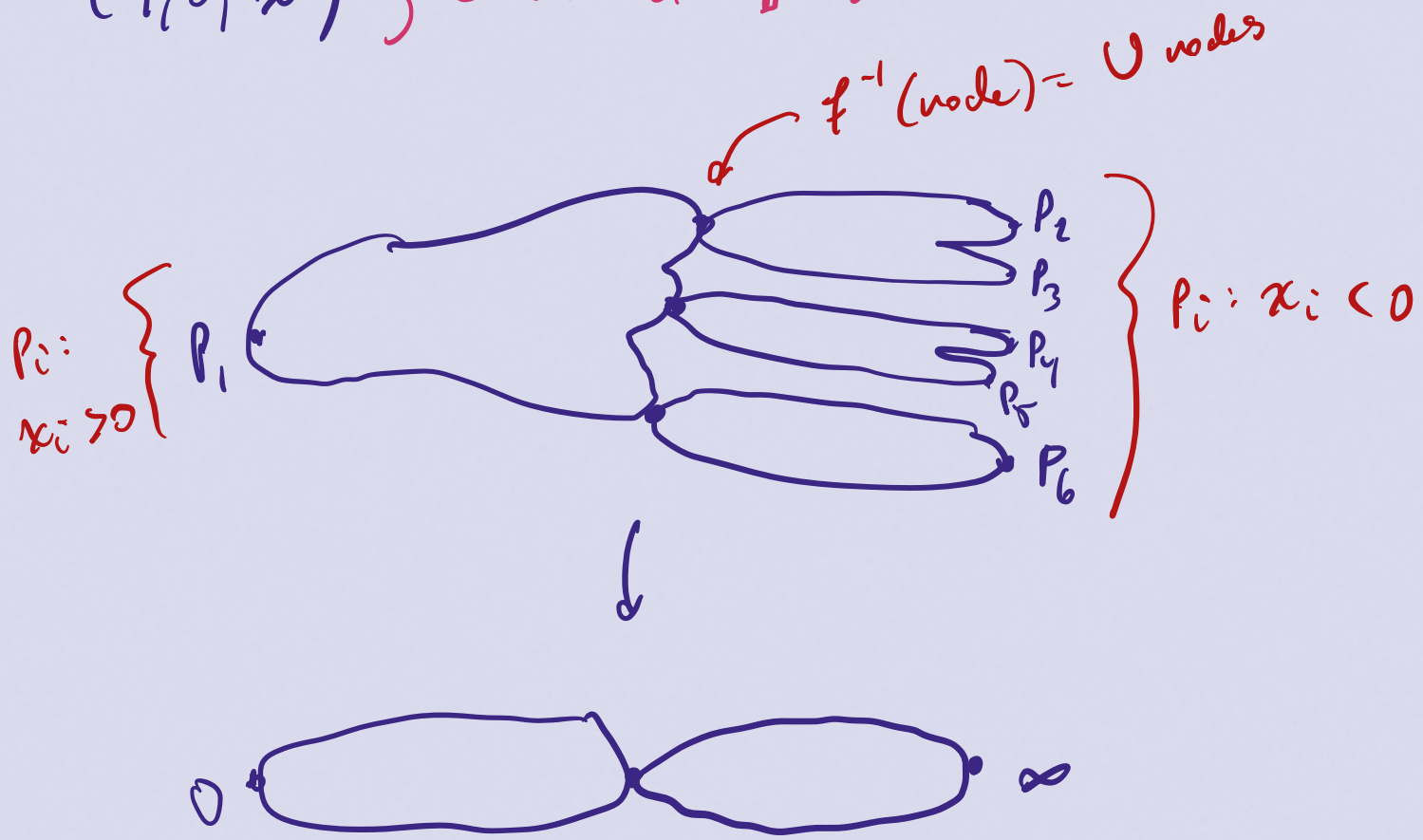


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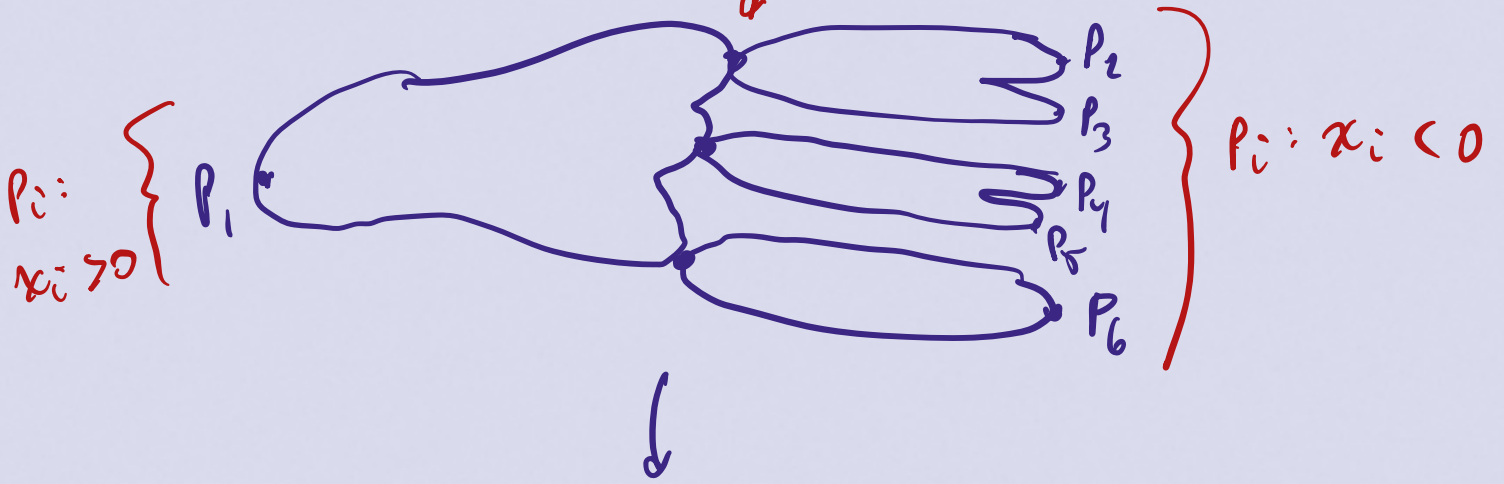
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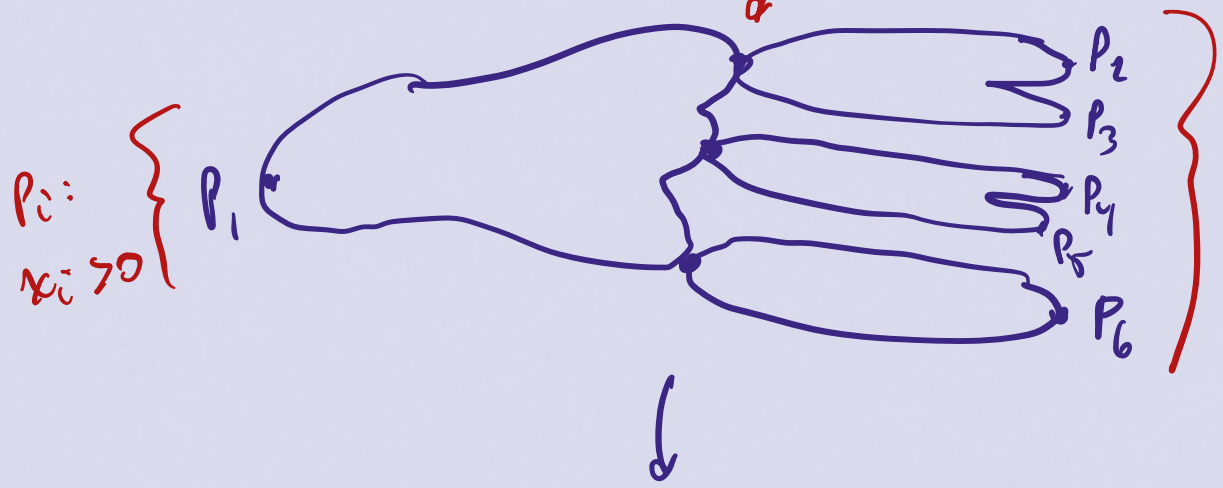
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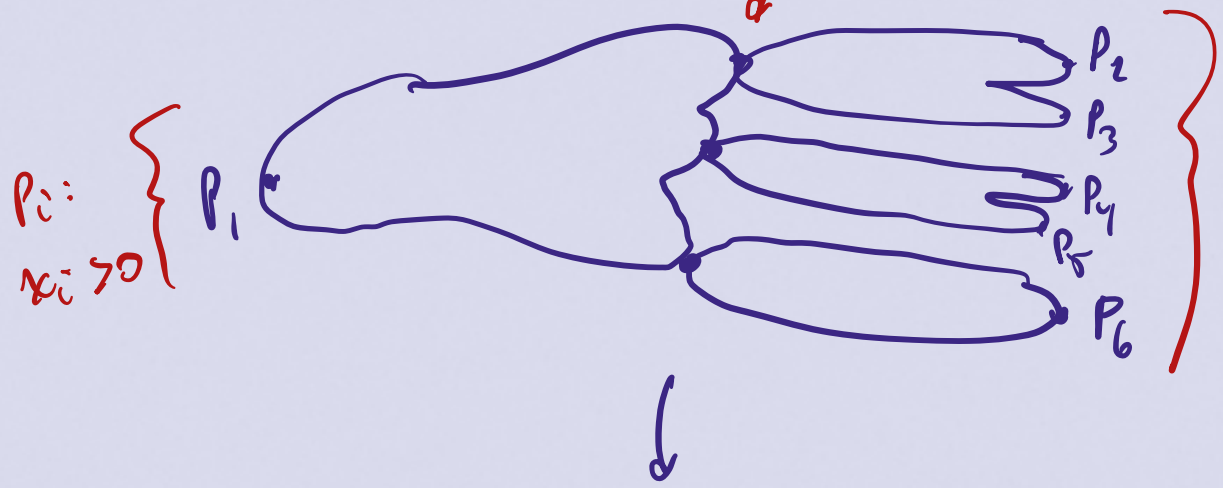
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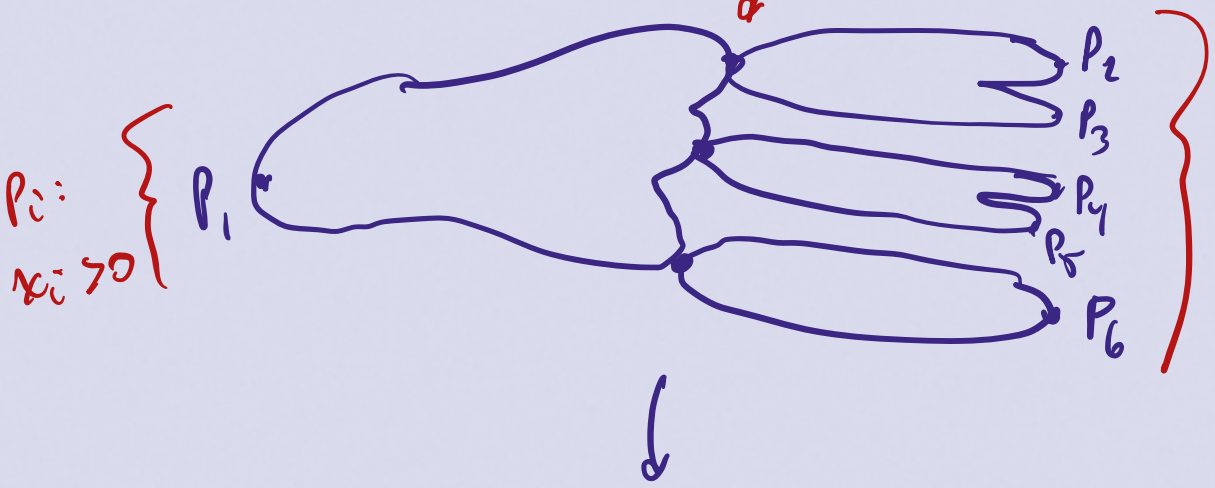
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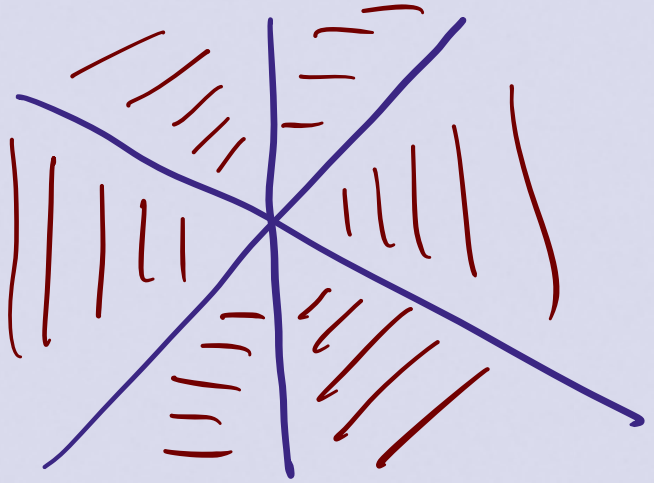
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e.g.: A_3



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Theorem (K):

As a function of $\vec{x}$, the class

$$[\mathcal{M}(\vec{x})] \in K_0(\mathrm{Var}/\mathbb{C})$$

is constant on the interiors of resonance chambers.

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explored further by CMJ + Ranganathan.

The space $\bar{M}_n$

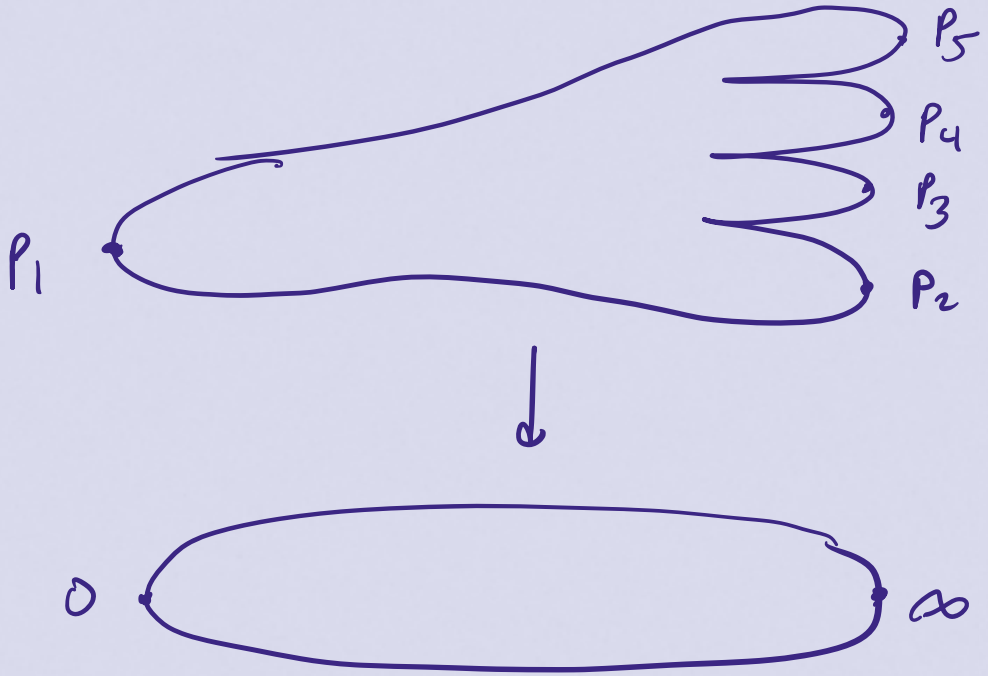
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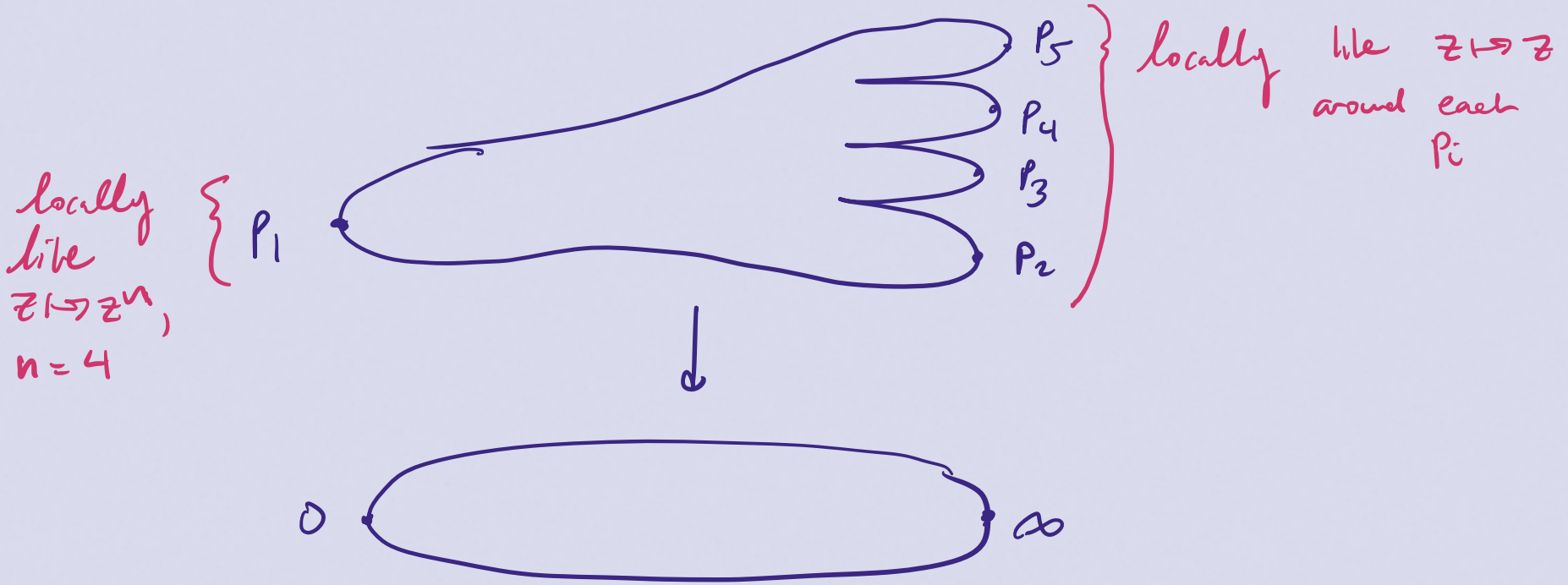
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n	$\chi(\bar{\mu}_{\text{Join}+1})$	$\chi(\bar{\mu}_n)$
2	1	1
3	2	2
4	7	10
5	34	84
6	213	1108
7	1630	20824
8	14747	530528

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$$\bar{\mathcal{M}}_n(K) := \pi_n^{-1}\left(\bar{\mathcal{M}}_{0,n+1}(K)\right) \subset \bar{\mathcal{M}}_n$$

Set $\Psi(s,t) := \sum_{n \geq 2} \sum_{k \geq 1} \chi(\bar{\mu}_n(k)) \frac{s^k}{k!} \frac{t^n}{n!} \in \mathbb{Q}[[s,t]]$

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Upshot: Recursive algorithm for computing the numbers $\chi(\bar{\mu}_n(k))$, and hence the numbers

$$\chi(\bar{\mu}_n) = \sum_{k=1}^{n-1} \chi(\bar{\mu}_n(k)).$$

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$$\nu(t) = t - \sum_{n\geq 2} \chi(\bar{\mathcal{M}}_{0,n+1}) \cdot \frac{t^n}{n!}$$

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Similar idea: Getzler–Pandharipande on $\bar{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$

Part II: getting our hands on $\vec{M}(\vec{x})$

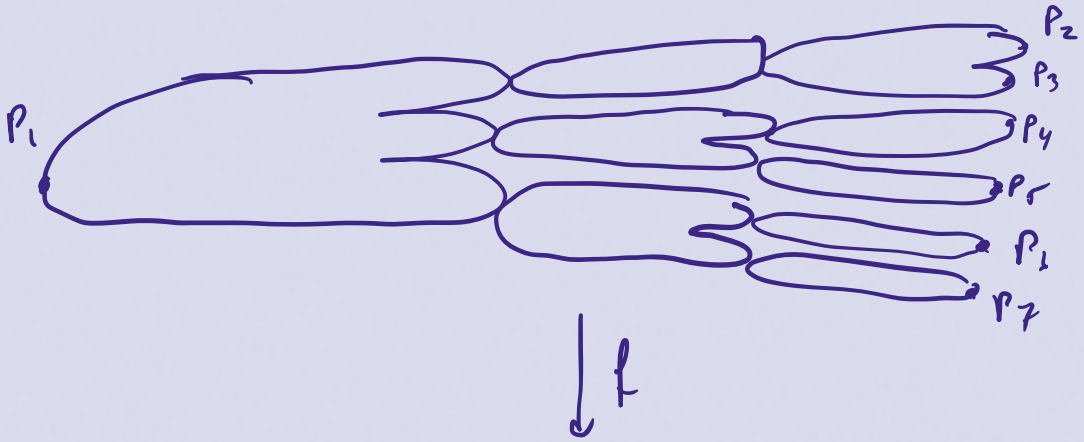
Stratification by Combinatorial type.

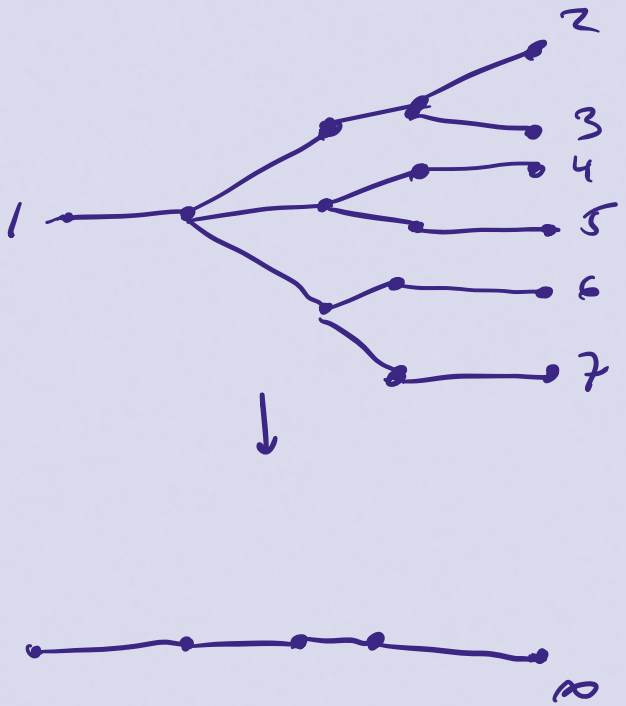
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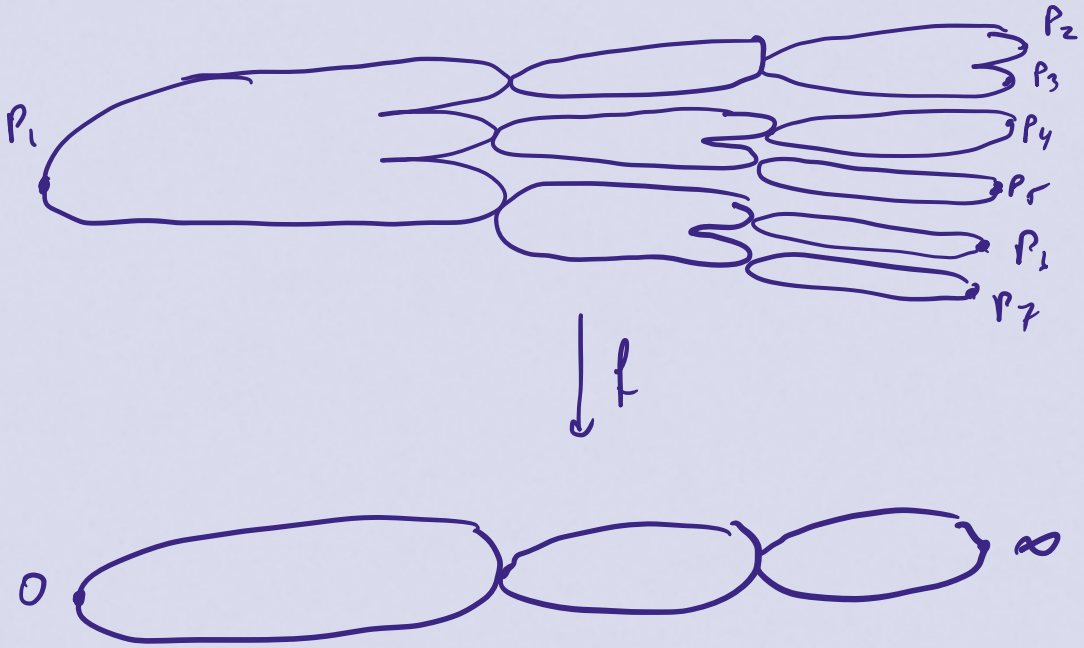
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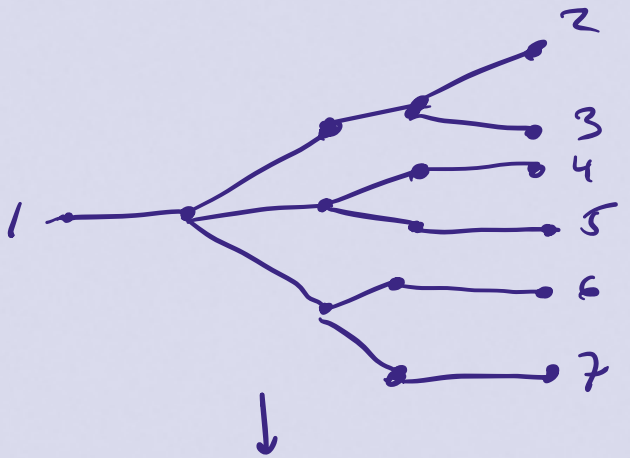


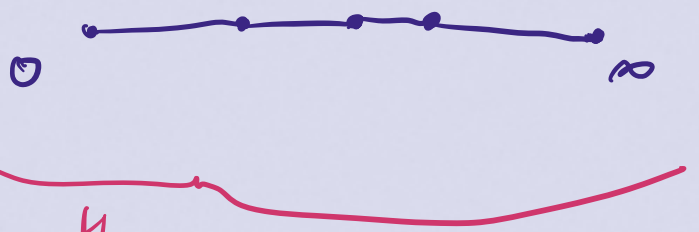


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This is the "combinatorial type" of the map

Let $\pi : T \to P$ be a combinatorial type.

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$$[\mathcal{M}(\vec{x})] = \sum_{\substack{\text{Combinatorial} \\ \text{types} \\ \pi : T \to P}} [\mathcal{M}(\pi : T \to P)]$$

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Goal: Show this sum only depends on the resonance chamber containing $\vec{x}$.

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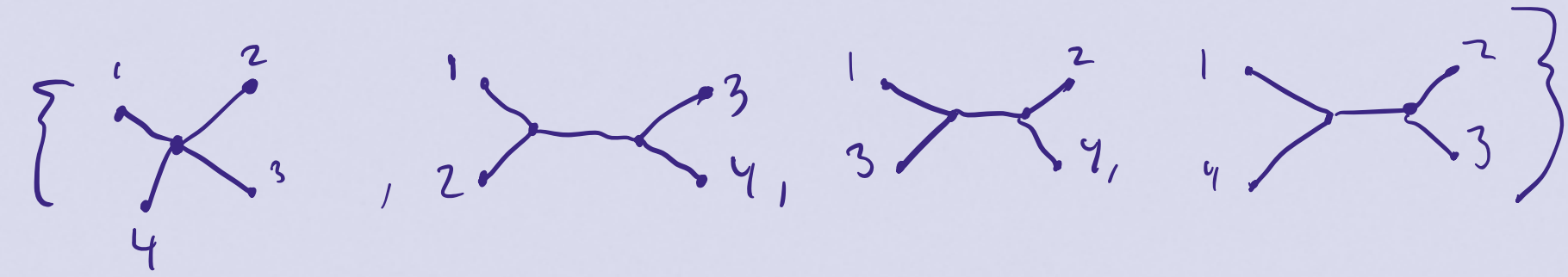
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eg:

$\Gamma_{0,4} =$ 

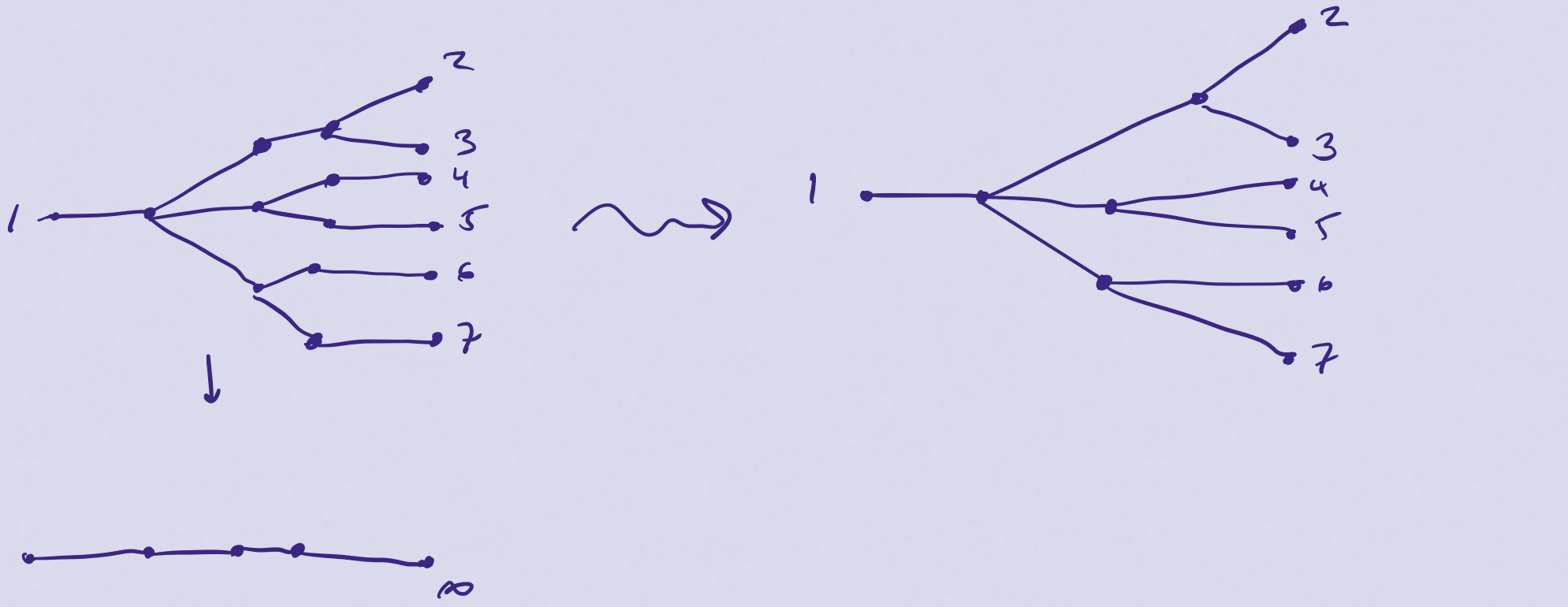
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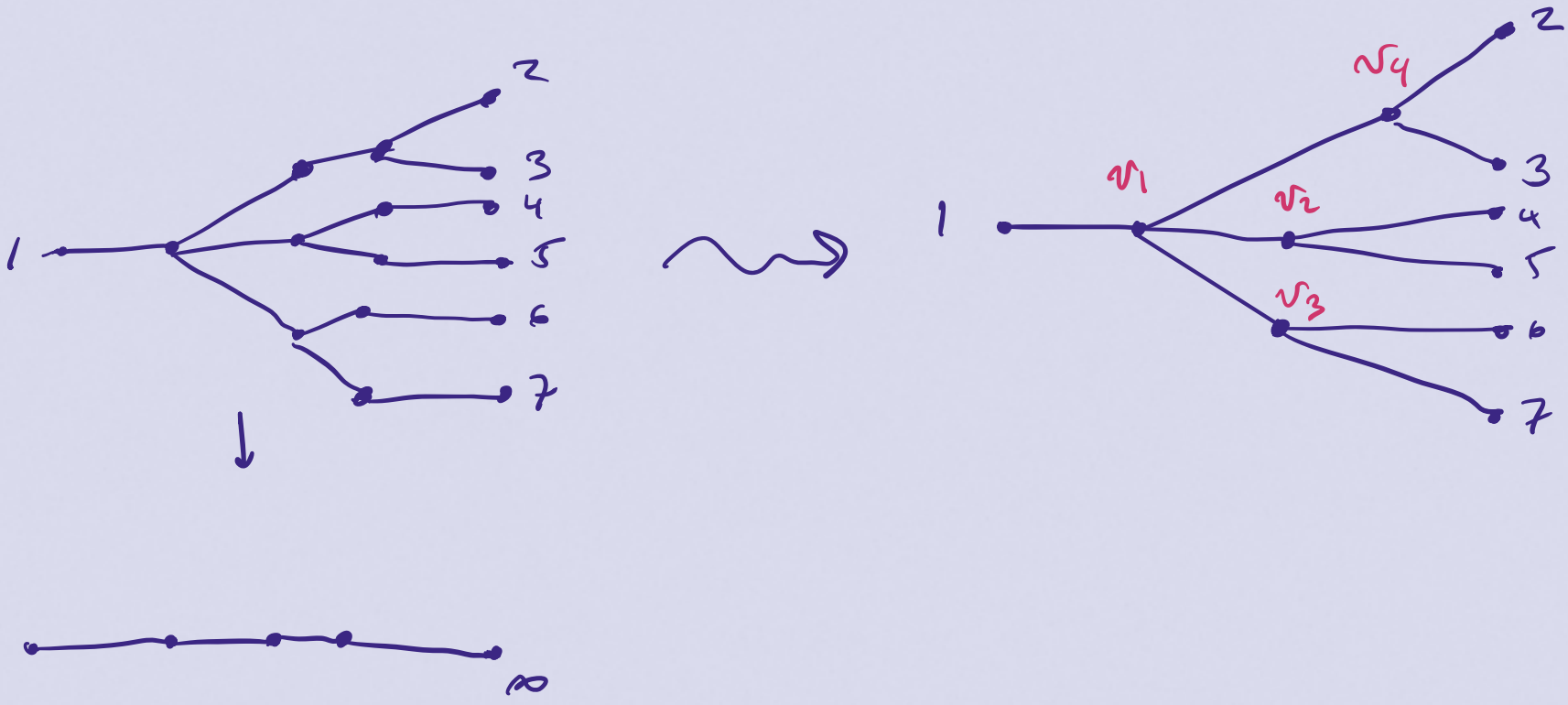
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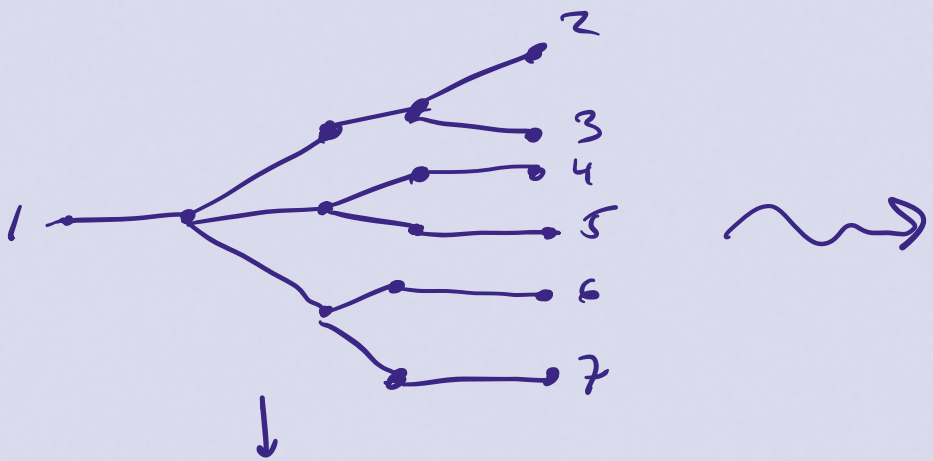
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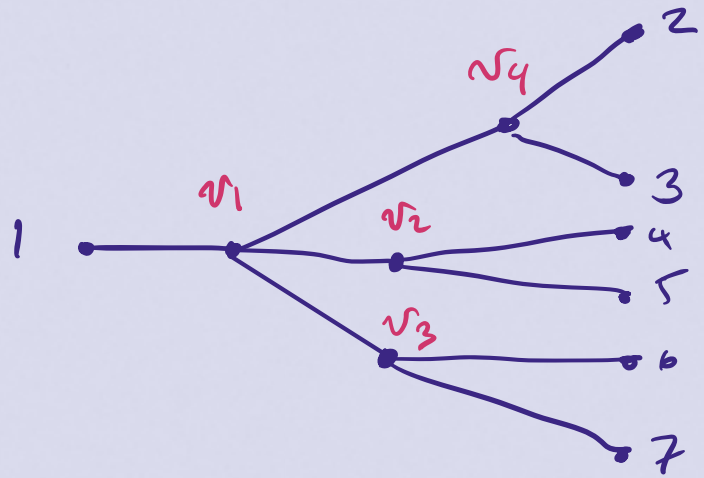


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Also get an ordered partition of the internal vertices of T!

$$\{v_1\} < \{v_2, v_3\} < \{v_4\}$$

Set $I(T) :=$ internal vertices of T.

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We have a map

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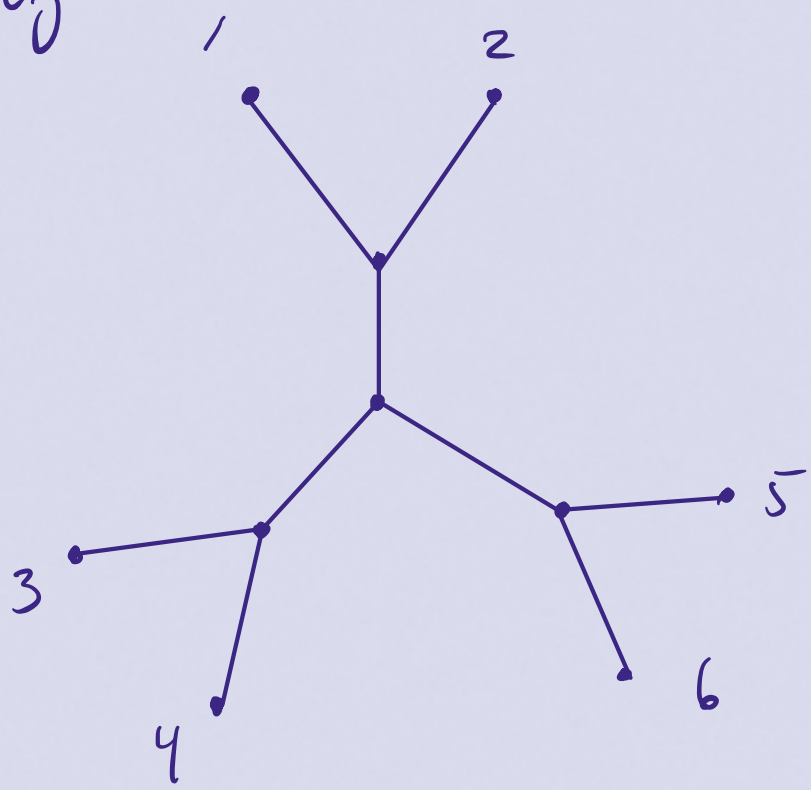
Question: Which ordered partitions arise in this way??

Directed tree structures.

Directed tree structures. Given $T \in \Gamma_{0,n}$, $\vec{x} \in A_n \cap \mathbb{Z}^n$, get a directed tree structure on T as follows:

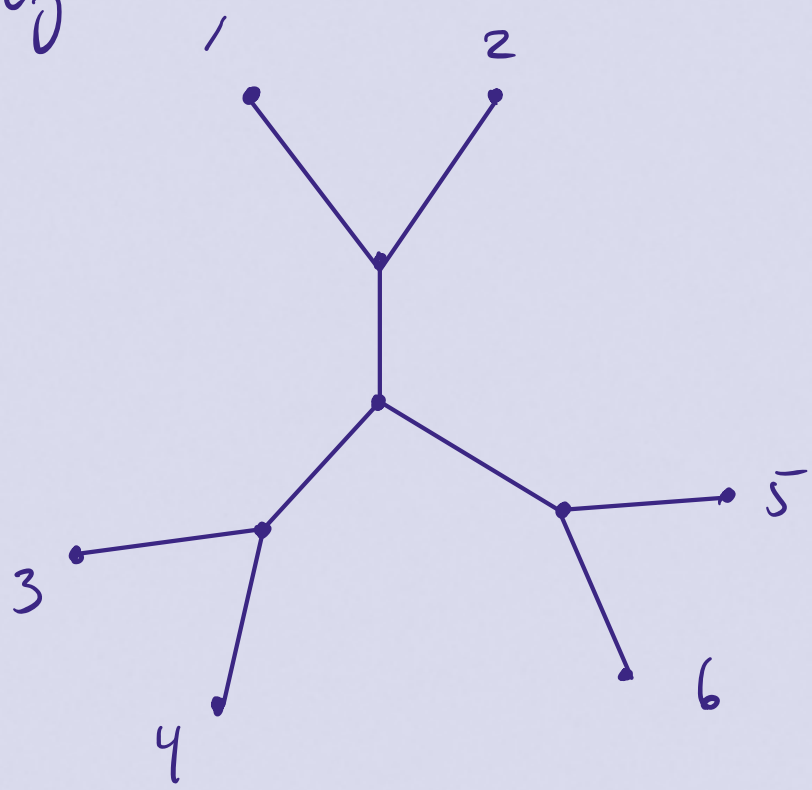
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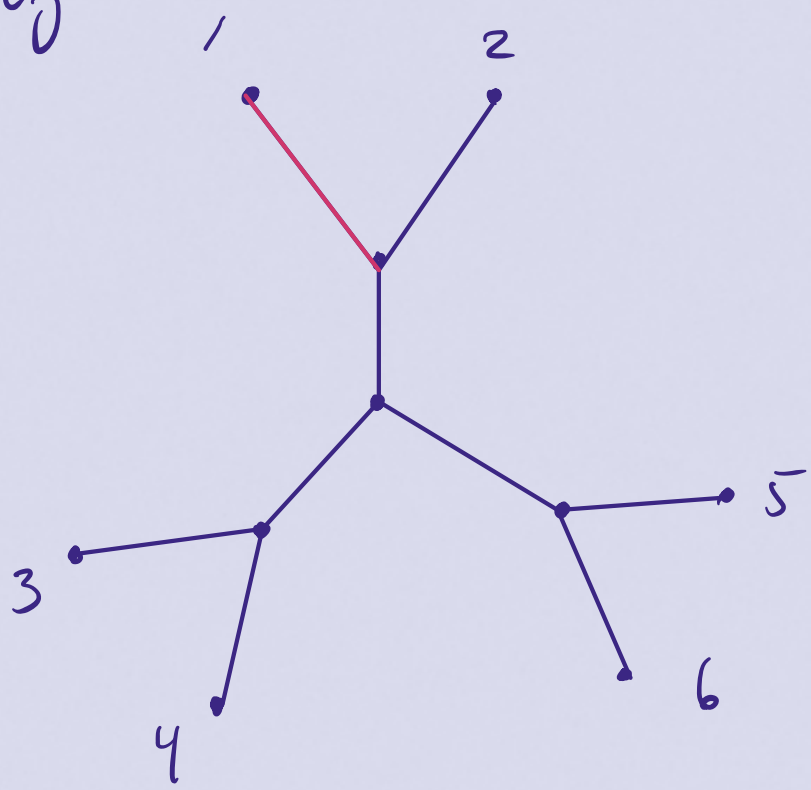
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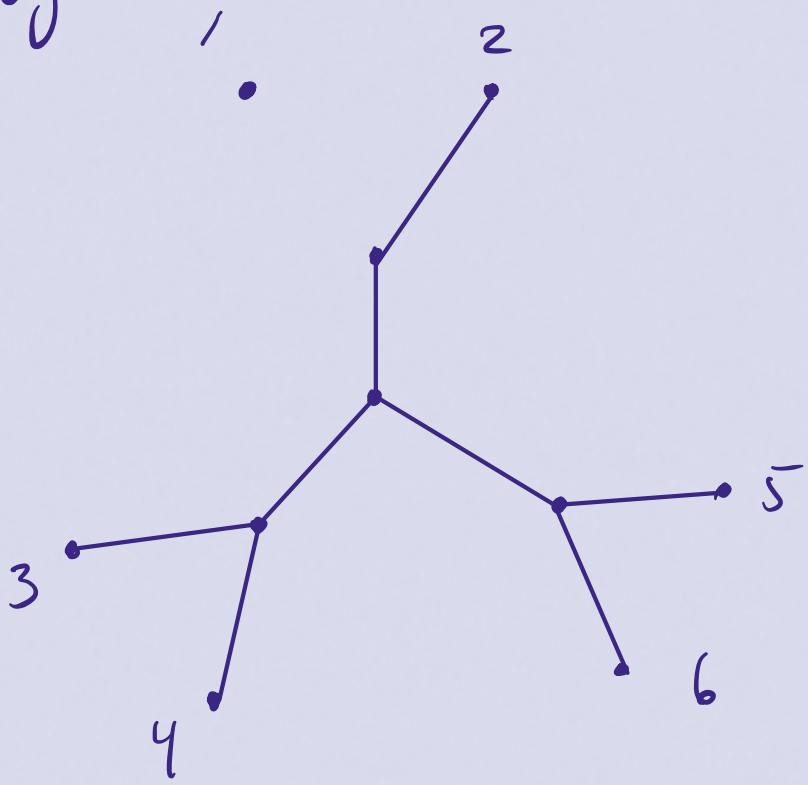
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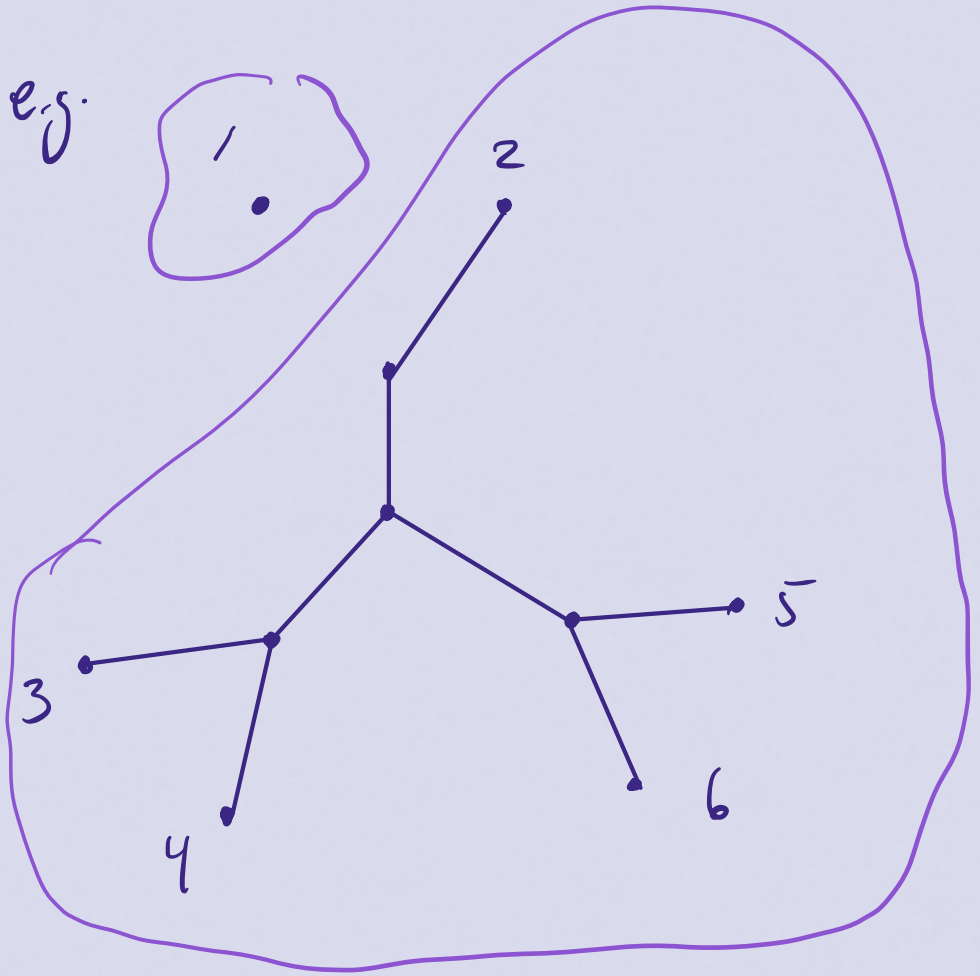


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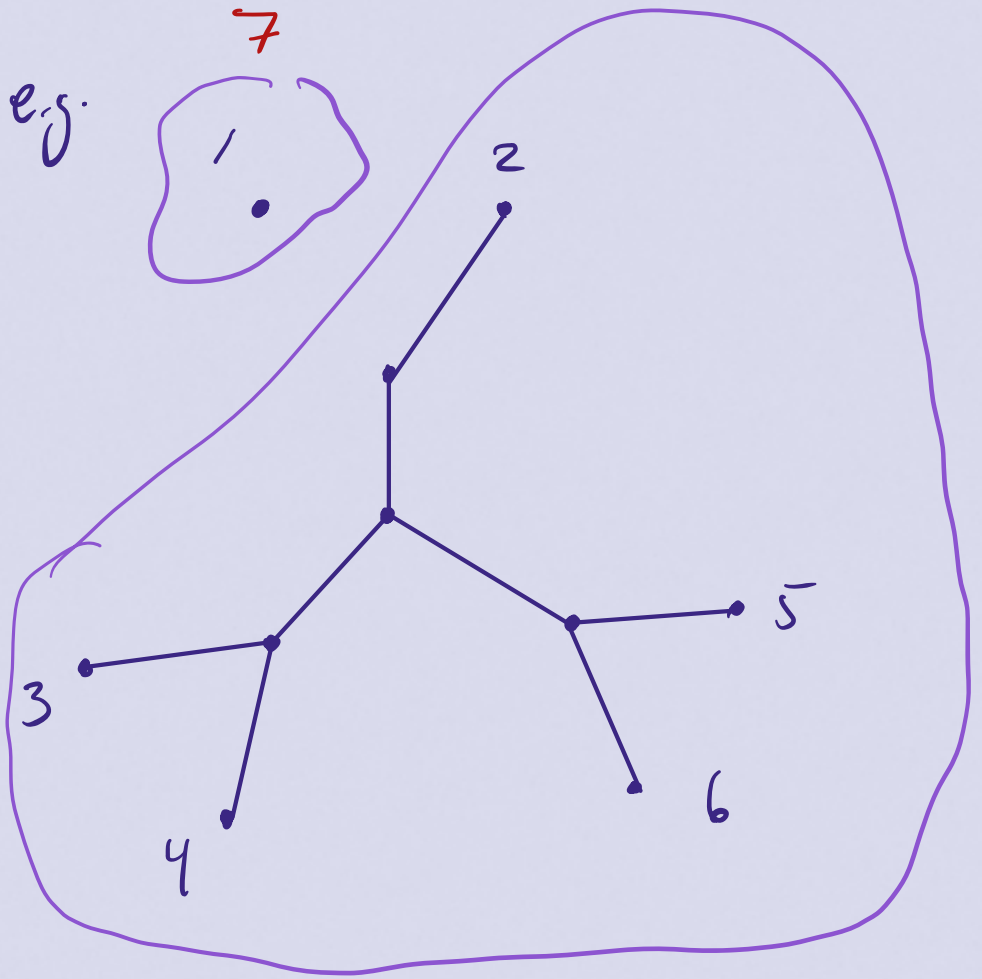
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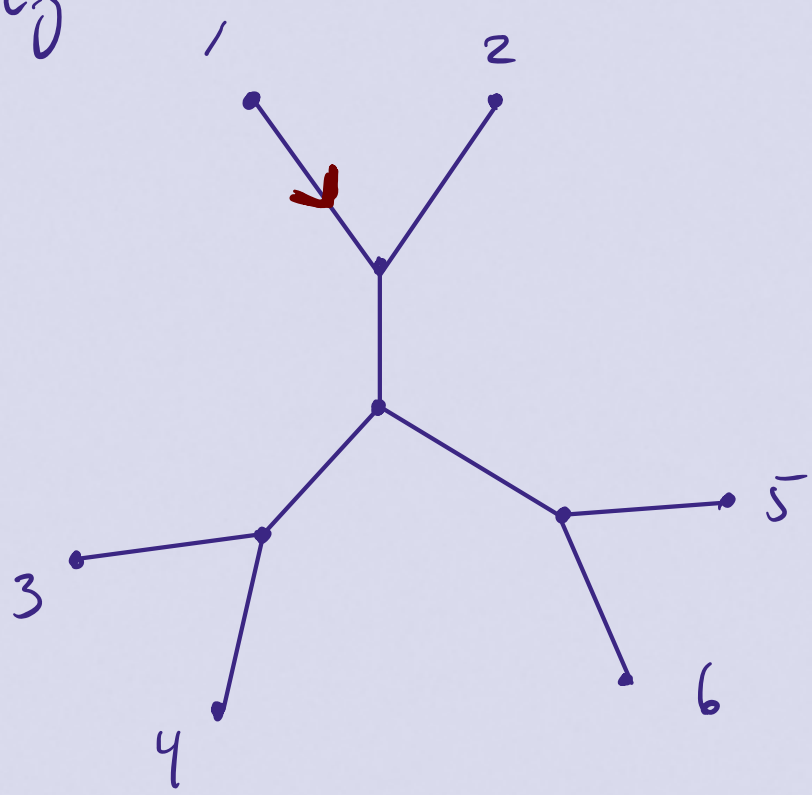
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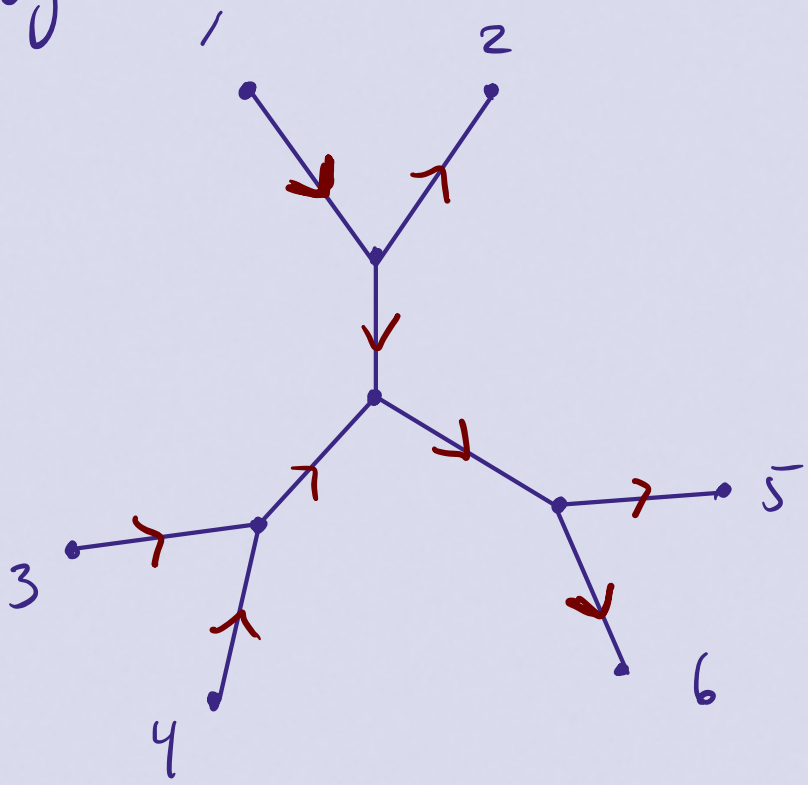
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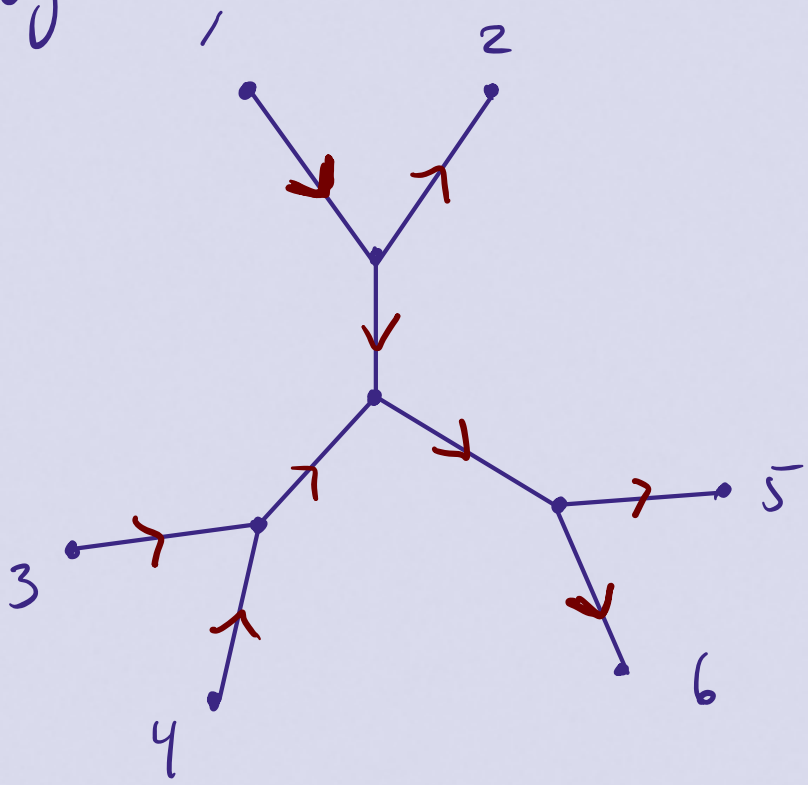
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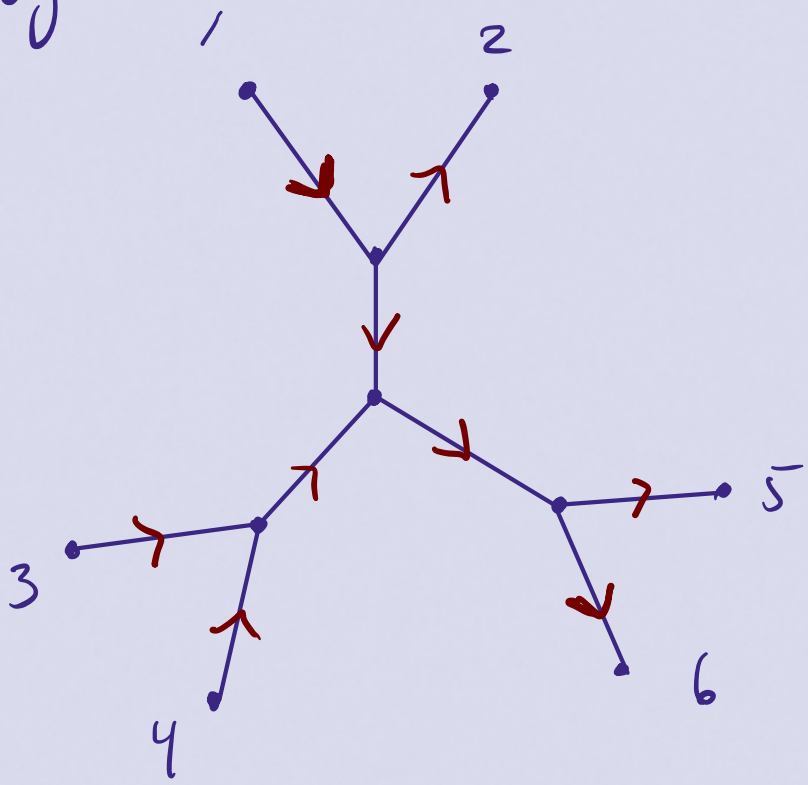
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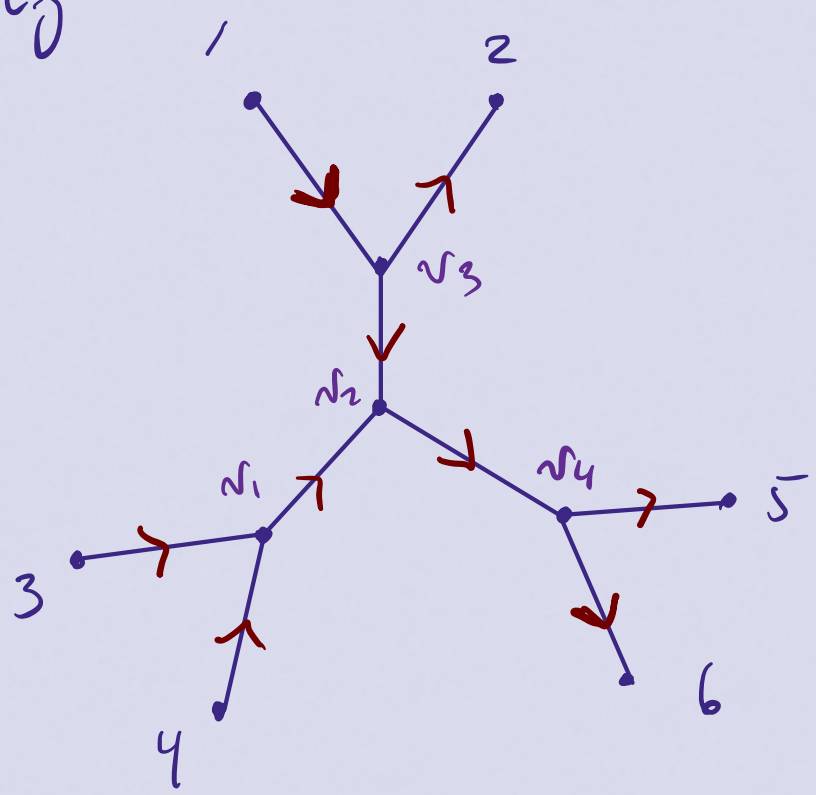
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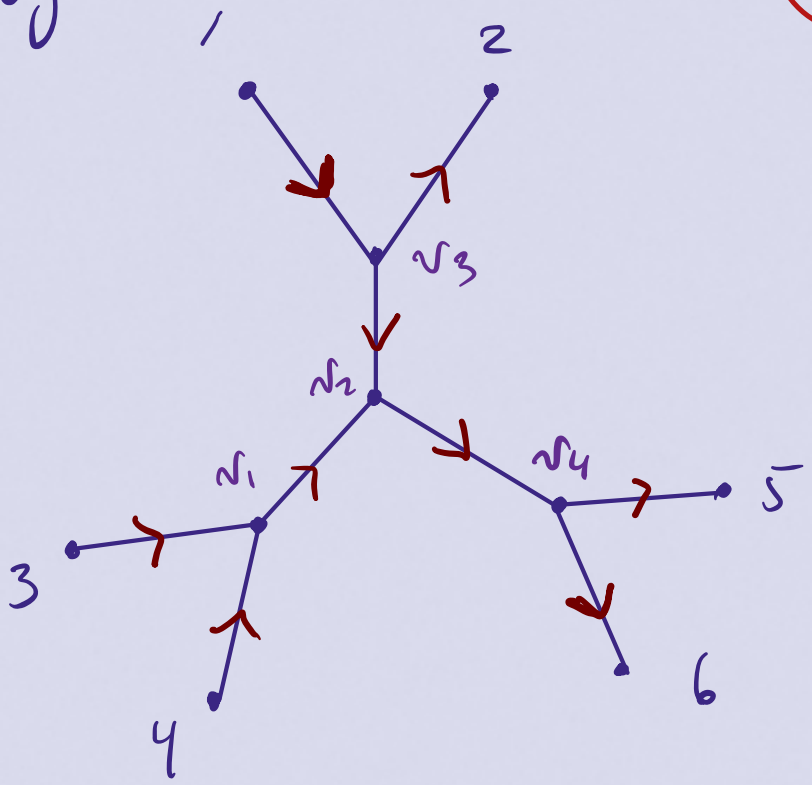
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The $\vec{x}$-directing depends only on the resonance chamber $\ni \vec{x}$!!

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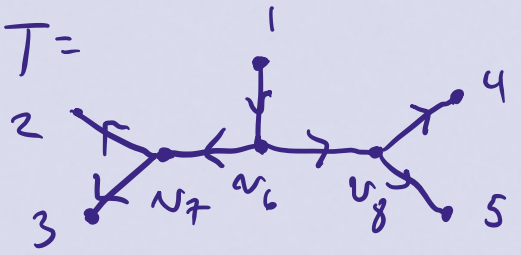
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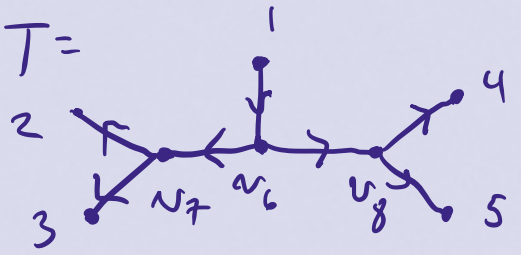


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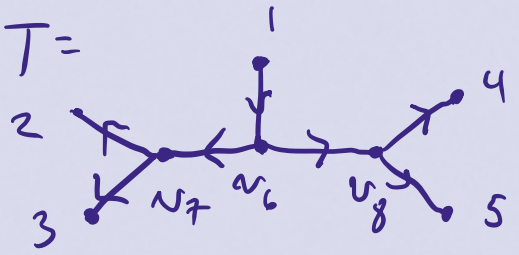
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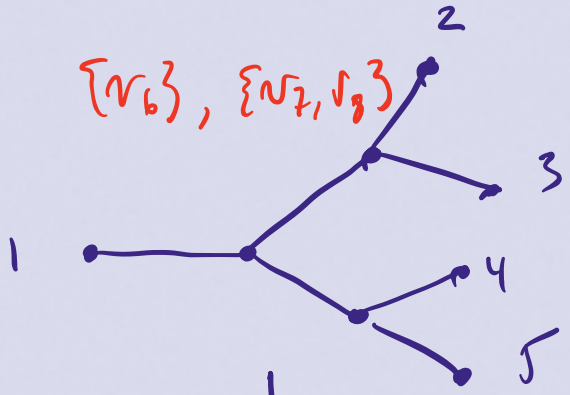
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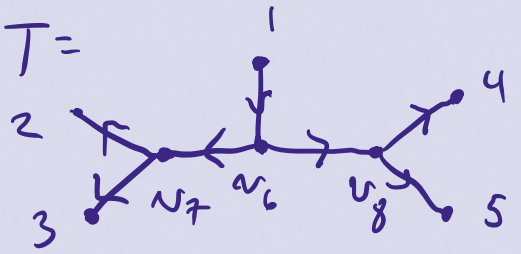


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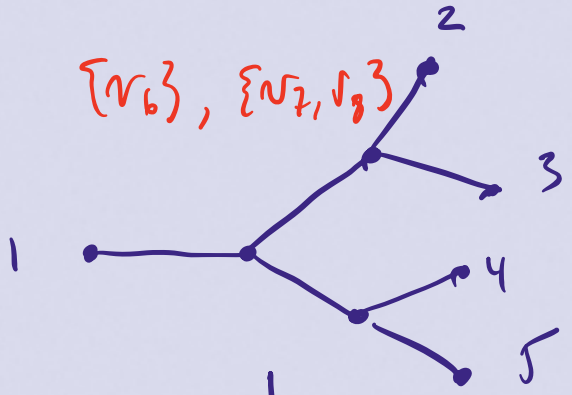
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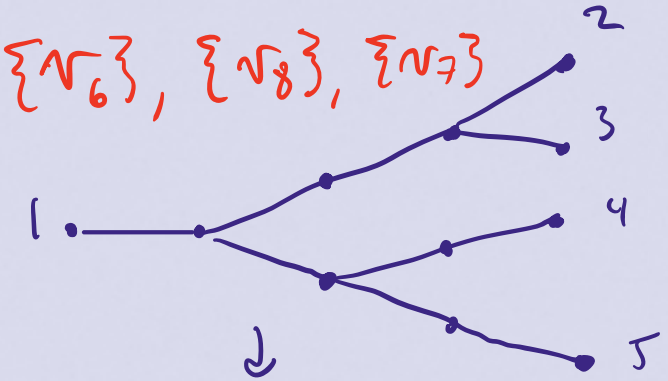
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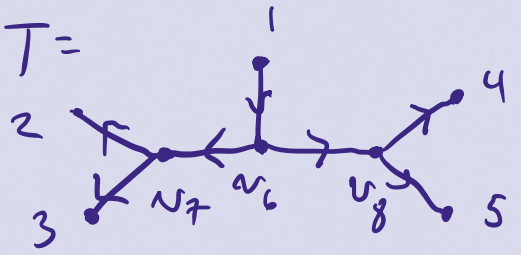




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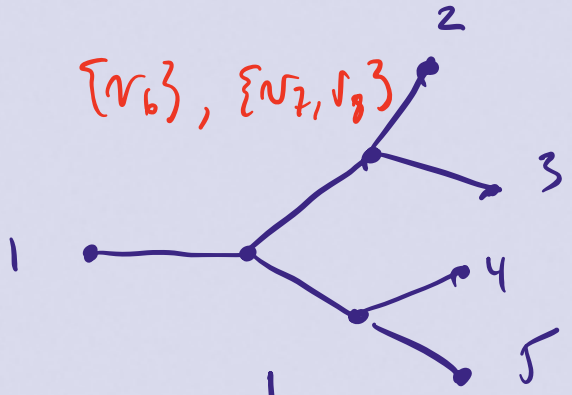
$$\left\{\begin{array}{l}\text{Combinatorial types}\\ \text{of maps in } \bar{\mathcal{U}}(\vec{x})\\ \text{whose stabilized source} = T\end{array}\right\} \longrightarrow \left\{\begin{array}{l}\text{ordered partitions}\\ \text{"refining" the}\\ \vec{x}\text{-directing of } T\end{array}\right\}$$

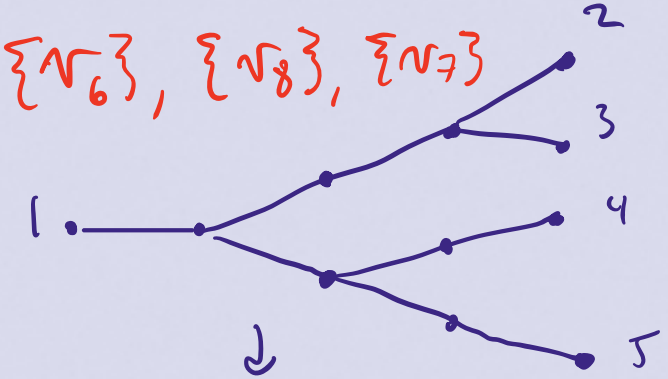
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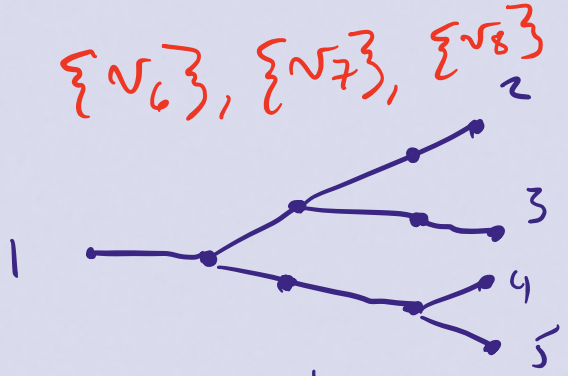
$T =$ 

$v_6 < v_7$

$v_6 < v_8$









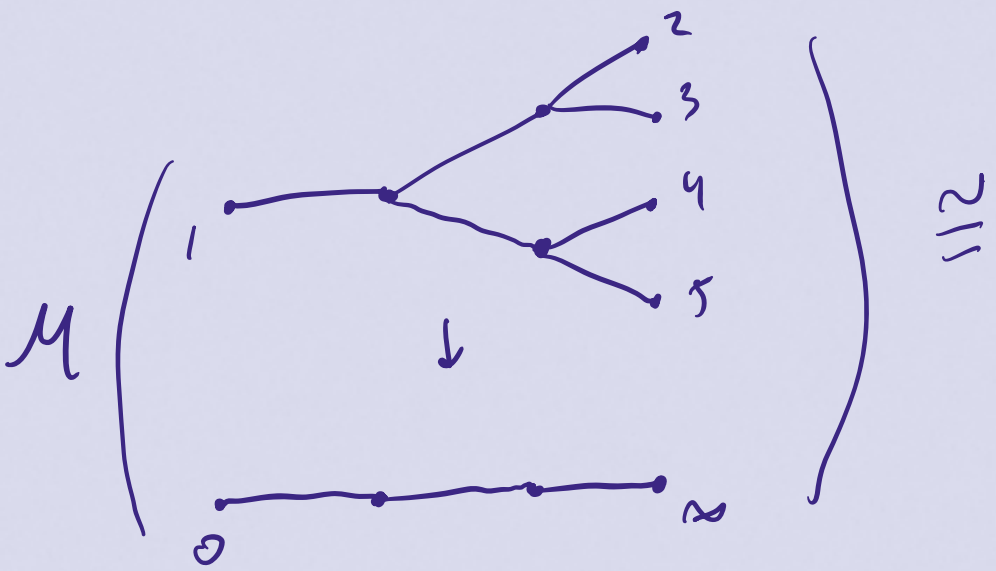




The stratum $\mathcal{M}(\pi : T \to P)$

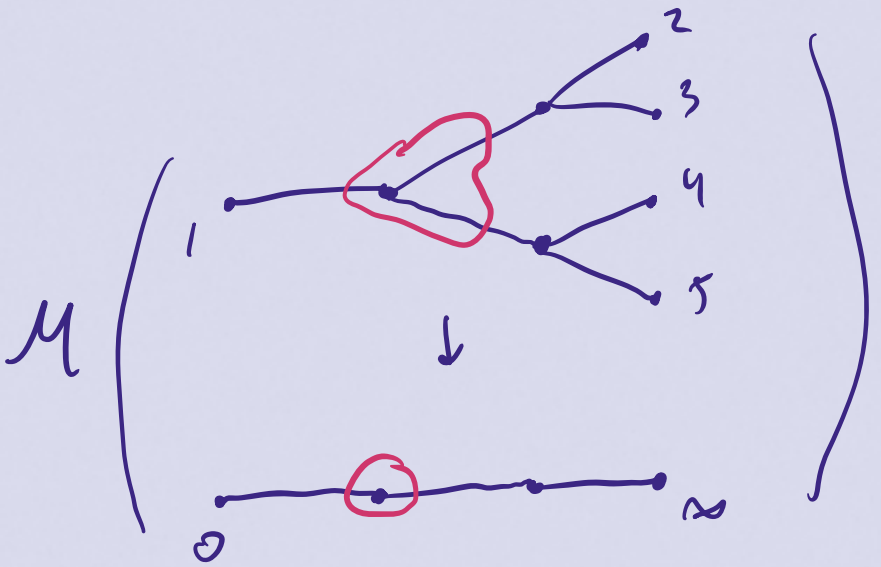
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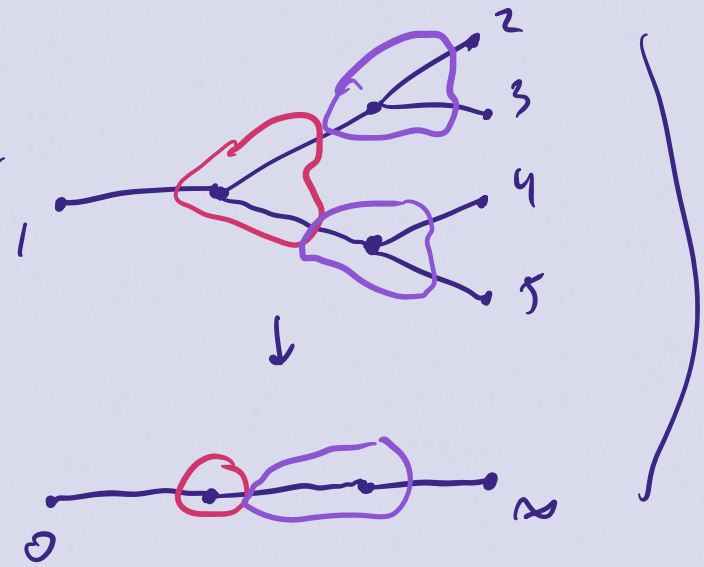


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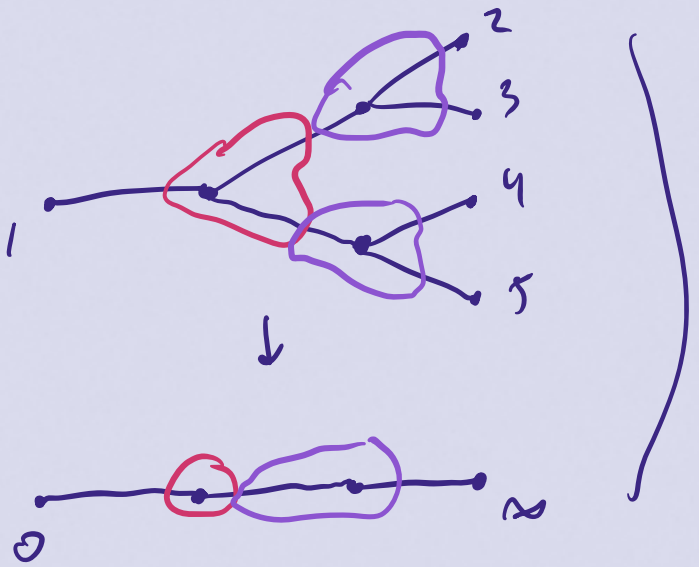
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$$\rightsquigarrow [\mathcal{M}_{0,3}]^3 [\mathbb{C}^*] \in K_0(\mathrm{Var}/\mathbb{C})$$

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Since $[\bar{\mathcal{M}}(\vec{x})] = \sum_T [\mathcal{M}_{\vec{x}}(T)]$, this shows $[\bar{\mathcal{M}}(\vec{x})]$ constant on resonance chambers.

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where $O_{\vec{x}}(T) = \#$ total orderings of $I(T)$ which refine $\vec{x}$-directing.

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Recall: $\mu_n := \bar{\mu}(\vec{x})$ for this $\vec{x}$

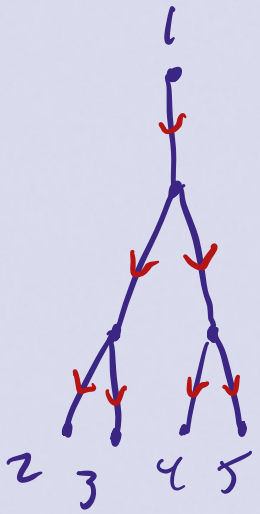
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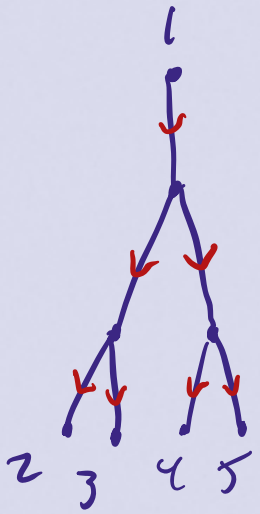
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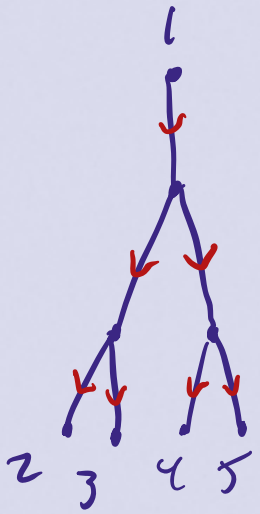


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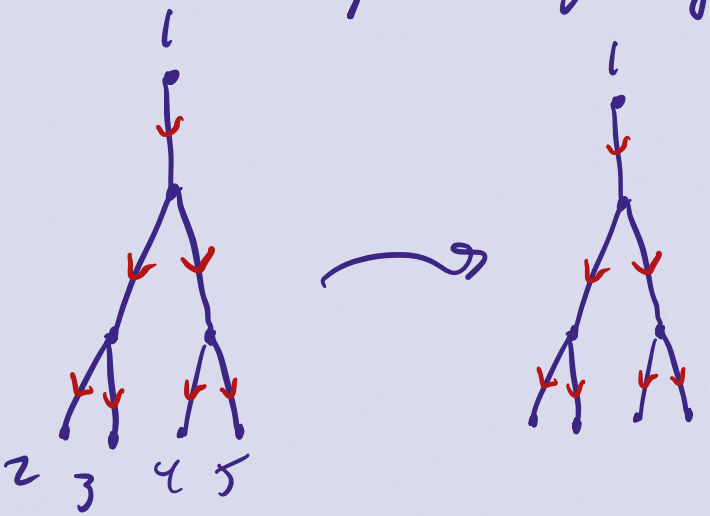
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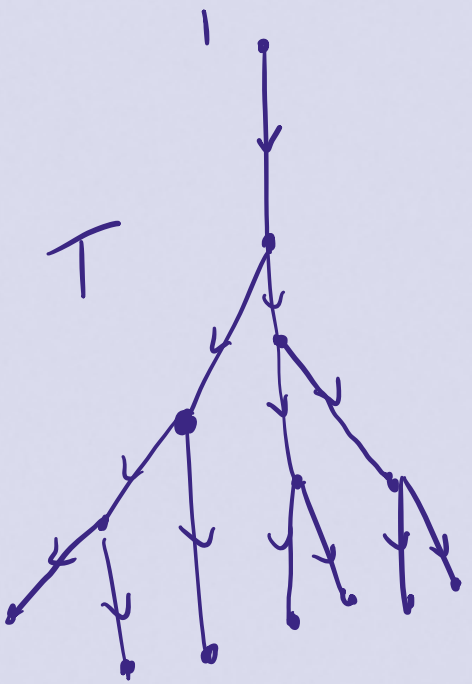
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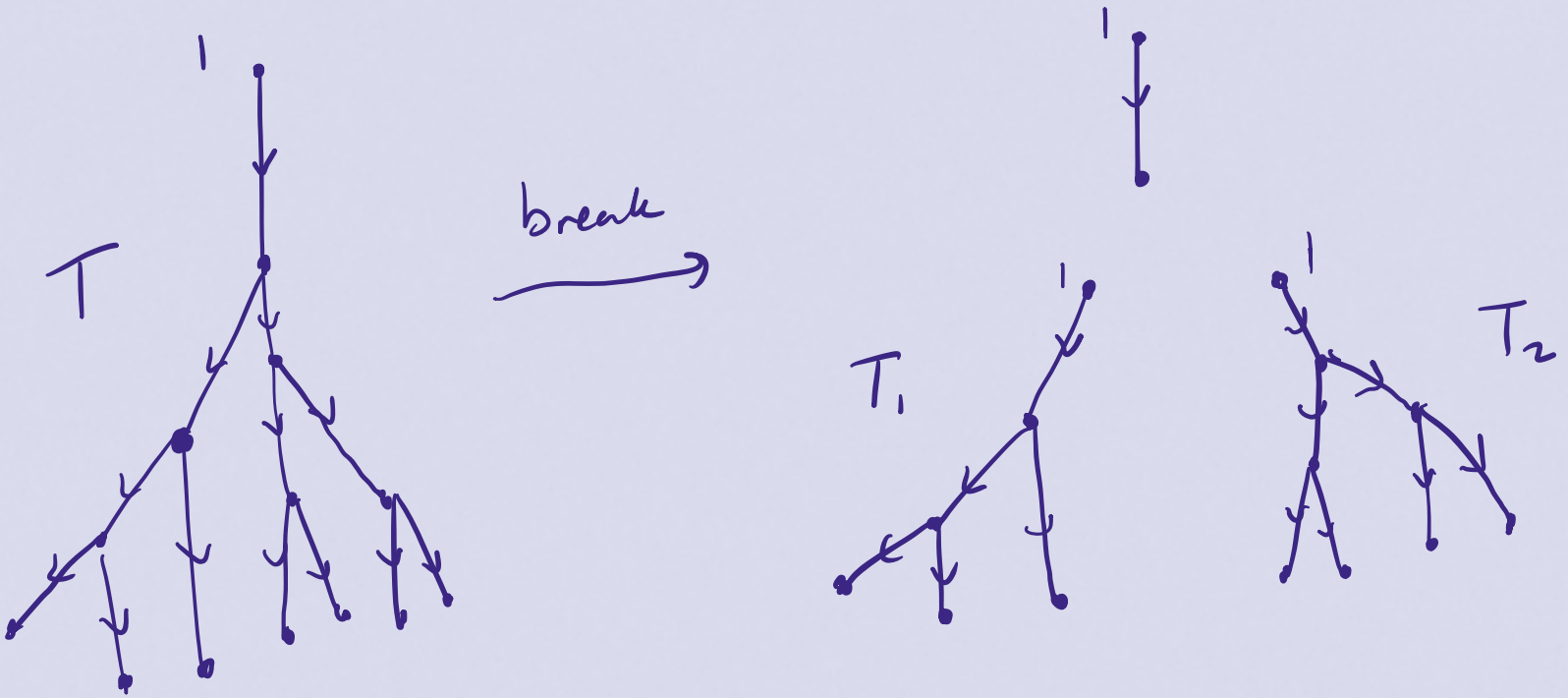


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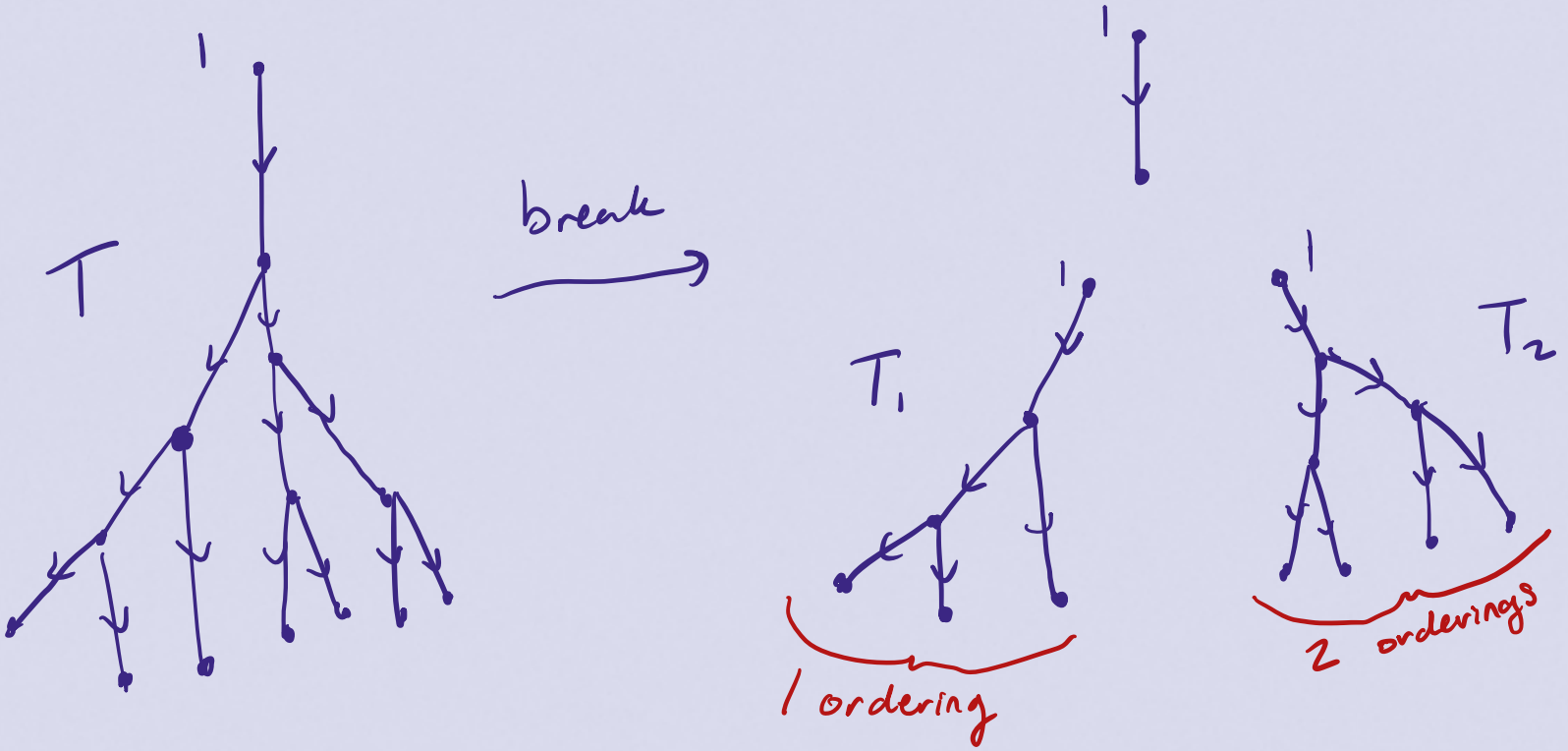
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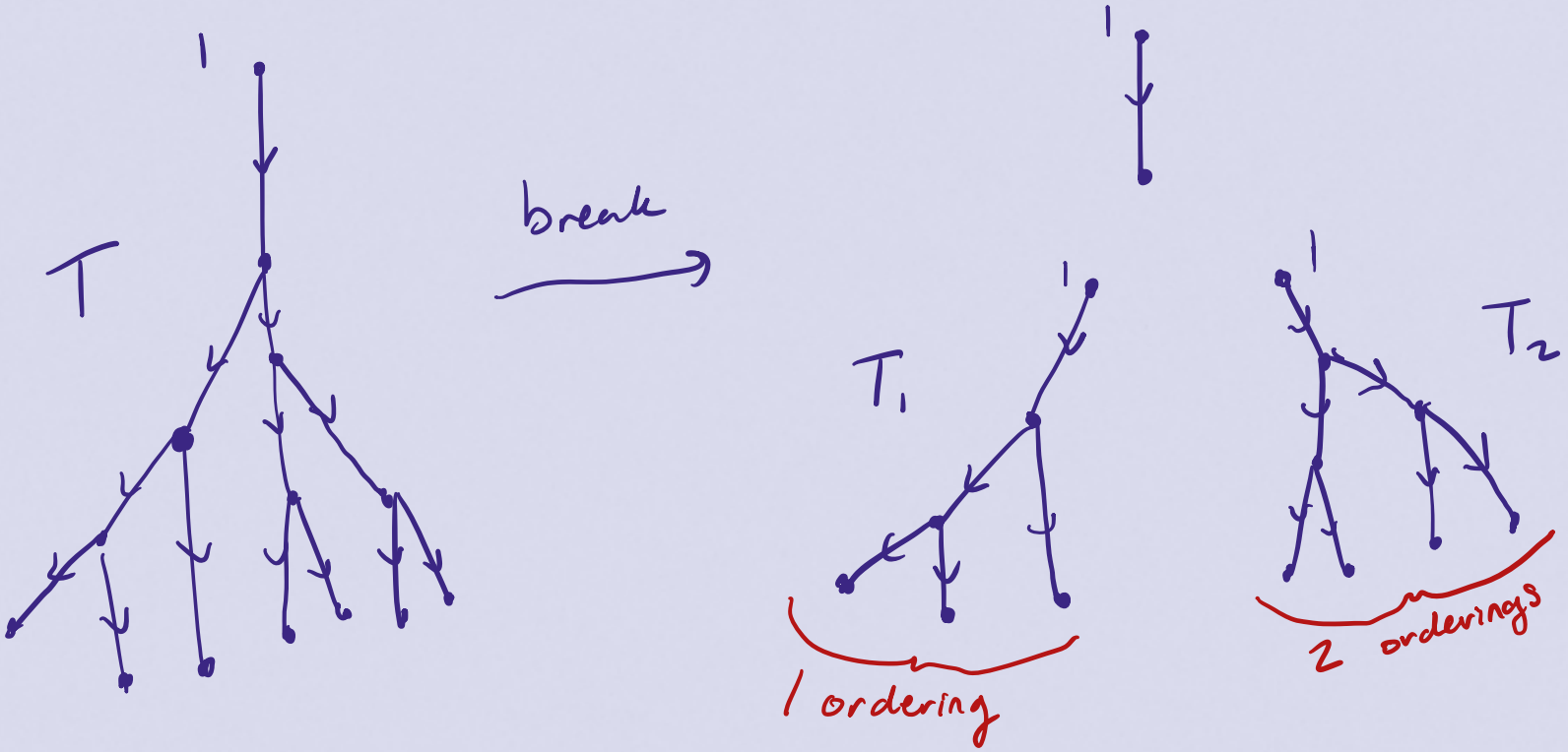
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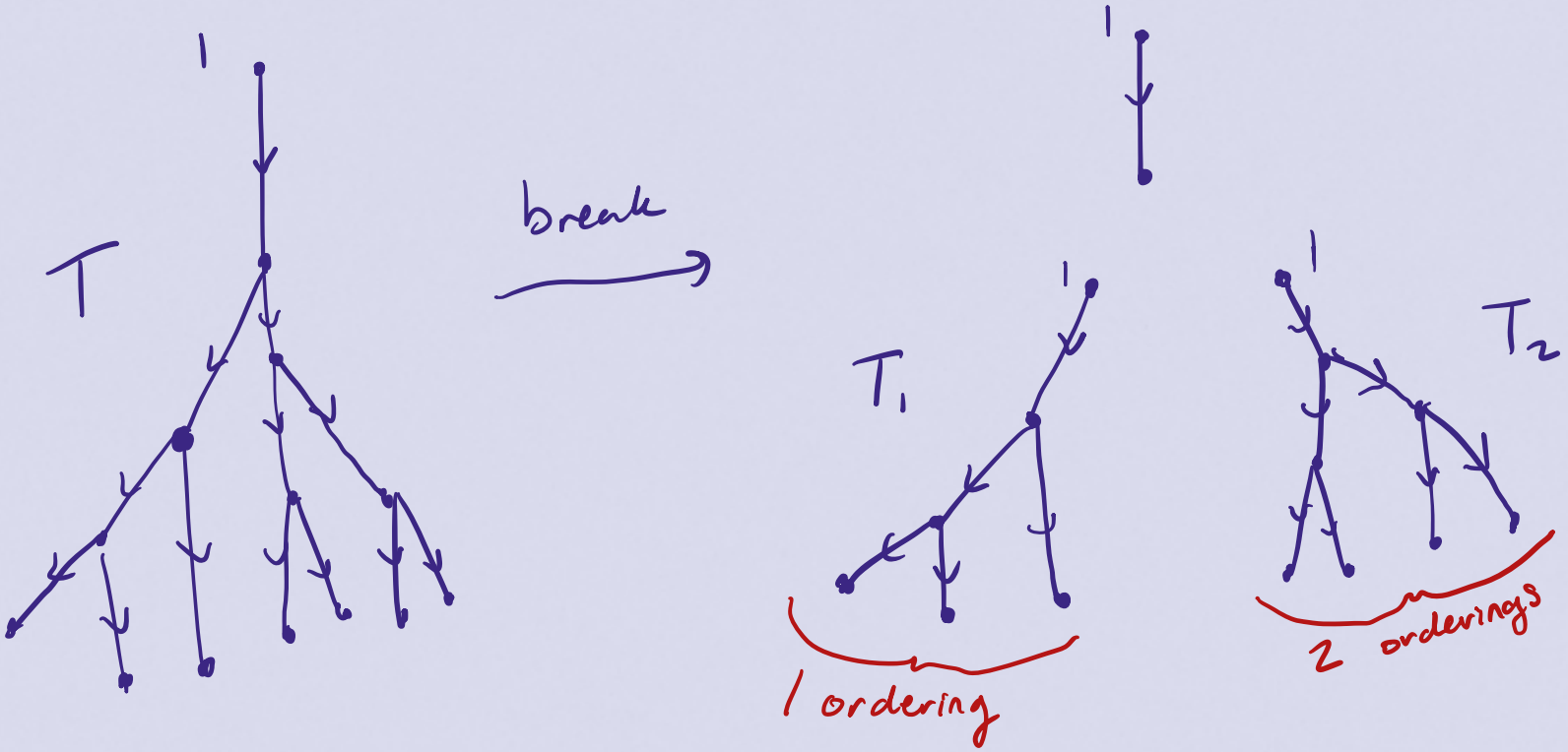
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$= 20.$

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$$v_m(t) = \sum_{j=1}^{m-1} v_1^{(j)}(t) \sum_{\substack{\lambda \vdash (m-1) \\ \ell(\lambda) = j}} \frac{(m-1)!}{\prod_{i=1}^{m-1} (i!)^{\lambda_i}} \prod_{i=1}^{m-1} \frac{v_i(t)^{\lambda_i}}{\lambda_i!}$$

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② Straightforward: $\nu_1(t) = (1+t)\log(1+t) - t$.

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(2) Equivariant Euler characteristic $\chi^{S_n}(\bar{\mathcal{M}}_n)$,

i.e. the Euler characteristic as a virtual S_n-representation.

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Conj: For a fixed homogeneous poly. $h(t_0, \dots, t_{n-3})$ of degree $n-3$, where t_i has degree i,

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is a piecewise poly. in $\vec{x}$.

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Conj: For a fixed homogeneous poly. $h(t_0, \dots, t_{n-3})$ of degree $n-3$, where t_i has degree i,

$$\int_{\bar{\mathcal{M}}(\vec{x})} h(c_0(\bar{\mathcal{M}}(\vec{x})), \dots, c_{n-3}(\bar{\mathcal{M}}(\vec{x})))$$

is a precewise poly. in $\vec{x}$.

TRUE for $h = t_1 t_{n-4}$