Properness of the K-moduli space
/$\mathbb{C}$

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Q Is there a $\underbrace{\text{"nice"}}_{\substack{\text{separated, proper} \\ \text{even proj.}}}$ moduli of $\underbrace{\text{Fano}}_{-K_X \text{ ample.}}$ varieties ?

Issue : $\mathcal{X} \to C \ni 0$ family of Fano s.t. $\mathcal{X}_t \cong X$ $(\forall t \neq 0)$

$\mathcal{X}_0 \ncong X$

e.g. $\mathbb{P}^2 \rightsquigarrow$ cone over conic $= \mathbb{P}(1,1,4)$

$\rightsquigarrow \mathbb{P}(a^2, b^2, c^2)$ $\quad a^2+b^2+c^2 = 3abc$ $\quad$ (Manetti)

$\mathcal{M}^{\text{Fano}}$ is "similar to $\mathcal{M}^{\text{v.b.}}$" : non-separated.

need stability condition

Kähler-Einstein metric (KE) $\quad Ric(g) = g$

- {Fano mfd w/ KE} is a metric space
  via (Gromov-Hausdorff distance b/t cpt metric space)

  $\to$ moduli of KE Fano var. should be separated.

- Donaldson-Sun : $\forall X_k$ = KE Fano mfd $(k=1,2,\cdots)$

  $\xrightarrow{GH} X_\infty$ = KE Fano var. (maybe singular)

  $\to$ moduli of KE Fano var. should be proper.

Yau-Tian-Donaldson conj. $\quad X$ = Fano var.

$\exists$ KE $\iff$ K-polystable (alg. condition)

K-moduli conj. $\exists$ proper moduli space of K-polystable Fano var

& all connected components are projective.

<u>Thm</u> (Alper, Blum, Codogni, Halpern-Leistner, Jiang, Li, Liu, Patakfalvi, Wang, Xu, —)

(1) Fix $n \in \mathbb{N}^*$, $v \in \mathbb{Q}_{>0}$. The moduli functor

$$\underset{\substack{\uparrow \\ sch/\mathbb{C}}}{\mathcal{M}^{Kss}_{n,v}(S)} = \left\{ \begin{array}{l} \text{flat family } \mathcal{X} \to S \text{ of K-ss Fano} \\ \text{var. w/ } \dim = n,\ vol = (-K)^n = V, \\ \text{satisfying Kollár's condition} \end{array} \right\}$$

is repn by an Artin stack of finite type.

(2) $\mathcal{M}^{Kss}$ admits a separated good moduli space M^{Kps} which parametrizes K-polystable Fano var.

[$\mathcal{M}^{Kss} \longrightarrow \underset{\substack{\uparrow \\ \text{alg. space}}}{M^{Kps}}$ étale locally look like $[\operatorname{Spec} A/G] \to \operatorname{Spec} A^G$]

(3) Every proper subspace of M^{Kps} that parametrizes KE Fanos is projective.

<u>Thm</u> (Liu, Xu, —) (1) M^{Kps} is proper
(2) YTD holds for any Fano var. $\Big\} \Rightarrow$ K-moduli Conj.

<u>Langton's algorithm</u> (for $\mathcal{M}^{v.b}$)

$C \ni 0$

$(\mathcal{E}_t)_{t \in C \setminus 0}$ s.s. v.b. $\to (\mathcal{E}_t)_{t \in C}$ v.b.

$\to$ if $\mathcal{E}_0$ is not s.s. $\quad \underset{\text{max destab.}}{F} \to \mathcal{E}_0 \to G$

$\to \mathcal{E}' = \ker(\mathcal{E} \to G)$, $\quad G \to \mathcal{E}_0' \to F$

stability improves.

$\to$ s.s. v.b.

<u>Plan</u>: • defn K-stability • max destab. for Fano.
• Langton's algorithm for $\mathcal{M}^{Fano}$

Defn " K-semistable = avg. div in $|-K_X|_{\mathbb{Q}}$ is log canonical "

- E = div / X : $E \subset Y \xrightarrow{\pi} X$ π : birat'l.
- $A_X(E) = 1 + \mathrm{ord}_E(K_{Y/X})$ log discrepancy.
- (X, D) is lc if $A_X(E) \geq \mathrm{ord}_E(\pi^* D)$ $\forall E/X$.
- $S_X(E)$ = expected vanishing order (" = ord_E(avg. div.)")
- K-ss $\Leftrightarrow$ $A_X(E) \geq S_X(E)$ $\forall E/X$
- m-basis type divisors : $D = \frac{1}{mN_m}\sum_{i=1}^{N_m} \{s_i = 0\} \sim_{\mathbb{Q}} -K_X$

  $s_1, \dots, s_{N_m}$ basis of $H^0(-mK_X)$
- $S_m(E) = \sup\{\mathrm{ord}_E(D) \mid D \text{ m-basis type}\}$

  $S_X(E) = \lim_{m\to\infty} S_m(E)$ (Blum - Jonsson)

Ex X = toric Fano $\rightsquigarrow$ $\Delta = \Delta(-K_X)$

K-ss $\Leftrightarrow$ barycenter of Δ is at the origin.

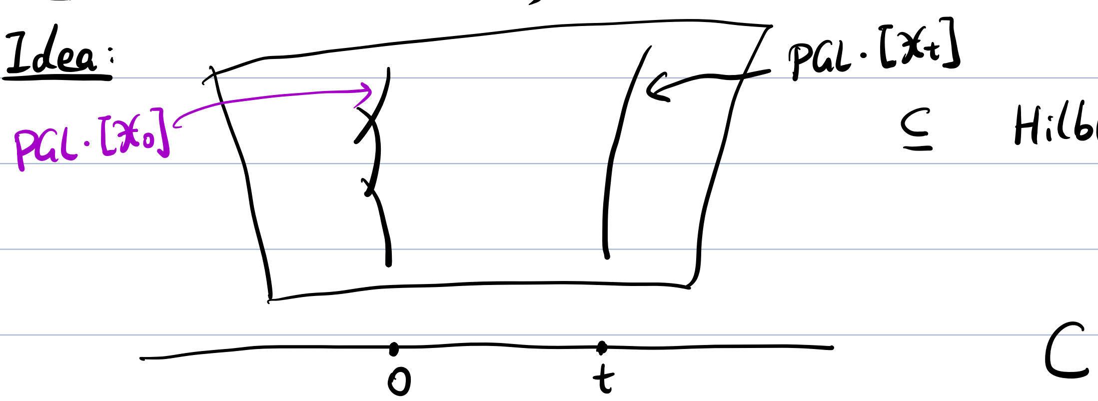
Defn (stability thresholds) $\delta(X) = \inf_{E/X} \frac{A_X(E)}{S_X(E)}$

(rmk : K-ss $\Leftrightarrow$ $\delta(X) \geq 1$)

Optimal destabilization conj. If X is not K-ss, then $\exists\, E/X$ s.t. $\delta(X) = \frac{A_X(E)}{S_X(E)}$

Thm (BHLLX, AHLH) ODC $\Rightarrow$ M^{Kps} is proper.

Idea:



$\subseteq \mathrm{Hilb}(\mathbb{P}^N) \times C$

($\mathcal{X}^0 \to C^0$ family of K-ss Fano $\longrightarrow$ $\mathcal{X} \to C$ family of Fano)

Blum - Liu - Zhou

If $\delta(X) = \frac{A_X(E)}{S_X(E)}$ then $E \rightsquigarrow$ degeneration $X \rightsquigarrow X_0$.

s.t. $\delta(X) = \delta(X_0)$

<u>BL</u>: $\delta(X)$ is lsc in family.

(compare: $F \to \mathcal{E}_0 \to G \xrightarrow{\text{degen}} F \oplus G \xrightarrow{\text{deform}} G \to \mathcal{E}_0' \to F$)

Remaining: ODC $\qquad \delta(X) = \frac{A_X(E)}{S_X(E)}$.

E_i / X limit ?

<u>Valuation</u> $\quad E/X \rightsquigarrow \mathrm{ord}_E : \mathbb{C}(X)^* \longrightarrow \mathbb{Z}$

$A_X(\cdot)$, $S_X(\cdot)$ extend to space of valuations $\mathbb{C}(X)^* \to \mathbb{R}$.

<u>Thm</u> (Blum - Jonsson) $\quad \delta(X) = \frac{A_X(v)}{S_X(v)}$ for some real valuation v.

$X \rightsquigarrow X_0$: if v is divisorial

$$\rightsquigarrow \mathcal{F}_v^\lambda H^0(-mK_X) = \{ s \in H^0(-mK_X) \mid v(s) \geq \lambda \}.$$

$$\rightsquigarrow X_0 = \mathrm{Proj}\Big(\underbrace{\bigoplus_{m,\lambda} \mathrm{Gr}^\lambda_{\mathcal{F}_v} H^0(-mK_X)}_{\text{finitely generated ?}} \Big)$$

$\mathrm{gr}_v R$

<u>Thm</u> (LXZ) If $\delta(X) = \frac{A_X(v)}{S_X(v)} \leq 1$ for some valuation v.

Then $\mathrm{gr}_v R$ is finitely generated.

$\big(\underset{<1}{\Rightarrow \text{ODC}}, \quad \underset{=1}{\Rightarrow \text{YTD}} \big)$

Odaka - Spotti - Sun: K-moduli for del Pezzo.

$X_3 \subset \mathbb{P}^4$, $X_3 \subseteq \mathbb{P}^5 \qquad$ Liu, Xu.