On the tangent space to the Hilbert scheme of points in $\mathbb{A}^3$

(joint w/ Alessio Sammartano)

<u>Notation</u> K alg. closed and $S := K[x_1, \dots, x_n]$.

<u>Def</u> • $H_n^d = \mathrm{Hilb}^d(\mathbb{A}^n) = \{ I \subseteq S : \ell(S/I) = d \}$

• $H_n^d \supseteq \mathcal{U} = \{ d\text{-distinct points in } \mathbb{A}^n \}$ and $\dim \mathcal{U} = nd$.

"Smoothable component"

	Smooth	Irreducible	Reducible	Singularities
H_1^d	$\simeq \mathbb{A}^d$ ✓	✓	—	—
H_2^d	✓ (Fogarty 1968)	✓	—	—
H_3^d	$d \leq 3$	$d \leq 11$ (Douvropoulos, Jelisiejew, Nødland, Teitler 2017)	$d \geq 78$ (Iarrobino 1984)	Is the singular locus of a hypersurface in a smooth variety (Dimca, Szendrői 2009)
H_4^d	$d \leq 3$	$d \leq 7$	$d \geq 8$	Expected (?) to be messy
$\vdots$				
H_{16}^d	$d \leq 3$	$d \leq 7$	$d \geq 8$	Satisfies Murphy's law up-to retraction (Jelisiejew 2020)

$I \subseteq S$, $T(I) = \mathrm{Hom}_S(I, S/I)$.

Q) For $I \in H_3^d$ how large is $\dim_K T(I)$?

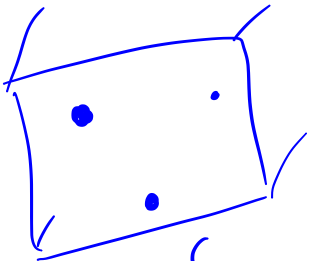
$\begin{pmatrix} \text{Diagonal} \\ \text{matices} \end{pmatrix} \subseteq \begin{pmatrix} \text{upper } \Delta \\ \text{matrices} \end{pmatrix} \subseteq GL(n)$ acts on H_n^d

$I \in H_3^d$

flat (in I)

I' torus-fixed / Borel-fixed

$\underbrace{\text{torus-fixed}}_{\text{monomial ideal}}$



A^3

I'' monomial ideal and supported at the origin.

Fact Monomial ideals lie in the smoothable component.

So $\dim T(I) \geqslant 3d$.

Ex $S = k[x,y,z]$, $m = (x,y,z)$, $\dim H_3^4 = 12$

$m^2 \in H_3^4$ and

$$\dim T(I) \leq \dim T(m^2) = 18$$

Conjecture [Briancon - Iarrobino]
Let $d = \ell(S/m^r)$ then

$$\binom{r+2}{3} \qquad \dim T(I) \leq \dim T(m^r) \quad \text{for all } I \in H_3^d.$$

Thm [R, S] Let $d = \ell(S/m^r)$. For all $I \in H_3^d$ we have

$$\dim T(I) \leq \frac{4}{3} \dim T(m^r).$$

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monomial ideal

$I \subseteq k[x_1, \dots, x_n]$ is $\mathbb{Z}^n$-graded

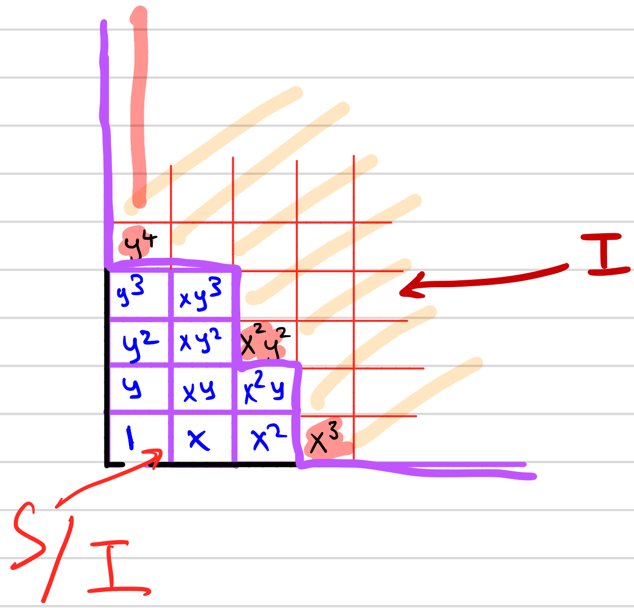
So $T(I) = \mathrm{Hom}(I, S/I)$ is $\mathbb{Z}^n$-graded

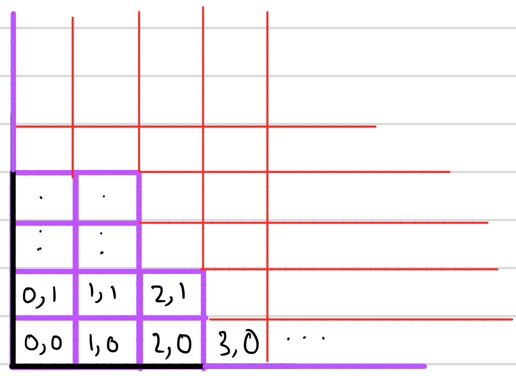
$$= \bigoplus_{\alpha \in \mathbb{Z}^n} T(I)_\alpha$$

Ex $I = (x^3, x^2y^2, y^4) \subseteq k[x,y]$

$$\ell(S/I) = \dim_k \operatorname{span}\{x^iy^j \notin I\}$$

Staircase





$x^iy^j \longleftrightarrow (i,j)$

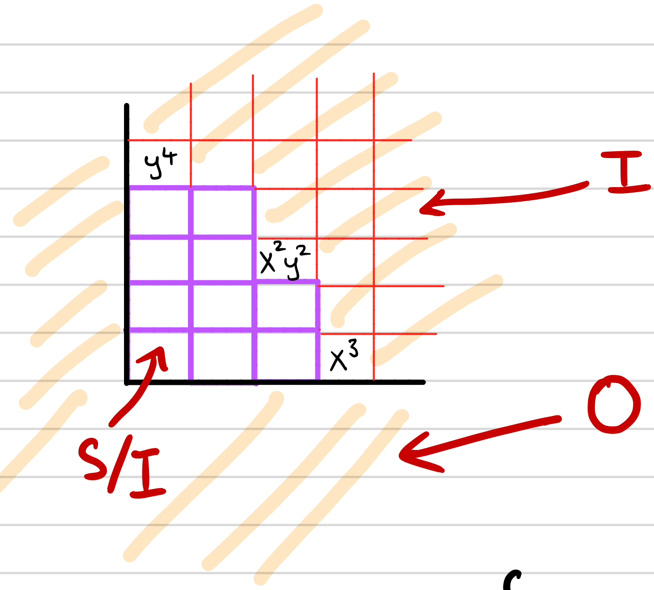
$\mathbb{N}^2 \longleftrightarrow$ Exponent vectors of monomials $\longleftrightarrow$ Point of $\mathbb{N}^2$ represented as a square.

$I = \bigoplus_{i \geq 0} (x^i y^{b_i})$ then $\ell(S/I) = \sum b_i$

finitely many non-zero b_i

$$I = (x^0y^4) \oplus (x^1y^4) \oplus (x^2y^2) \oplus (x^3 \cdot y^0) \oplus \cdots$$

Ex $\quad I = (x^3, x^2y^2, y^4)$, $\ell(S/I) = 10$, $\dim T(I) = 20$



Squares correspond to points in $\mathbb{Z}^2$

$\mathrm{Hom}(I, S/I)$

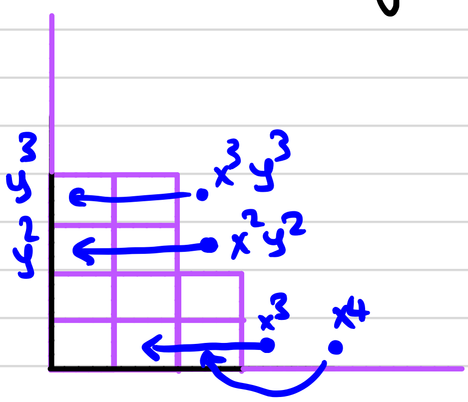
Some tangent vectors:

$x^3 \mapsto x$

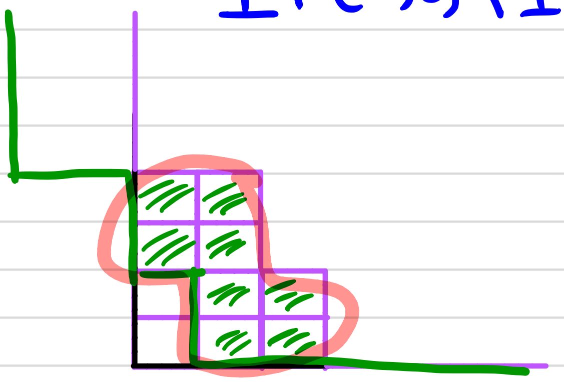
$x^2y^2 \mapsto y^2$

$y^4 \mapsto 0$

degree $(-2,0)$



$\tilde{I} + (-2,0) \setminus \tilde{I}$



All the monomials with non-zero image:

$x^2y^3 \mapsto xy^2 \qquad x^3y^3 \mapsto xy^3$

$x^2y^2 \mapsto y^2 \qquad x^3y^2 \mapsto xy^2$

$x^3y \mapsto xy \qquad x^4y \mapsto x^2y$

$x^3 \mapsto x \qquad x^4 \mapsto x^2$

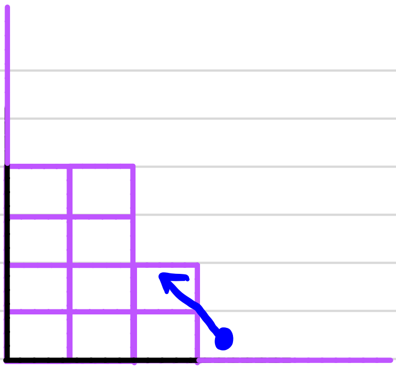
How do we encode all of this combinatorially?

Defn: Let $\tilde{I} = \{(\alpha_1, \dots, \alpha_n) : x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in I\} \subseteq \mathbb{N}^n$

$x^3 \mapsto x^2y$

$x^2y^2 \mapsto 0$

$y^4 \mapsto 0$



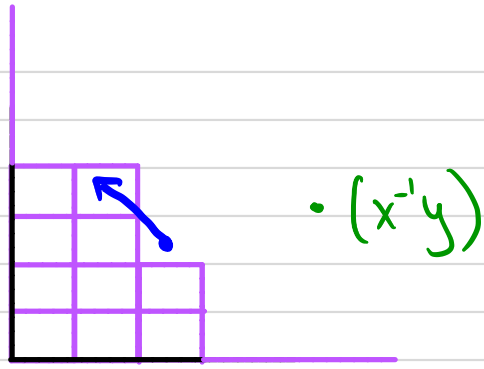
degree $(-1,1)$

$x^3 \mapsto 0$

$x^2y^2 \mapsto xy^3$

$y^4 \mapsto 0$

degree $(-1,1)$

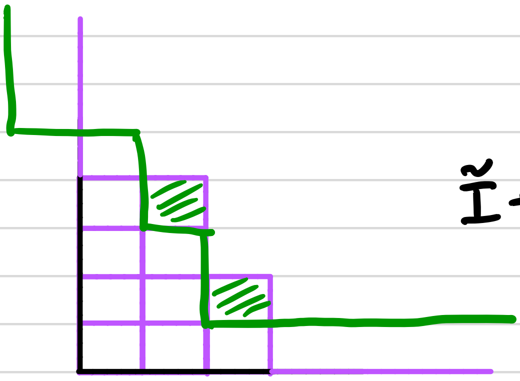


Non-zero images:

$x^3 \mapsto x^2y$

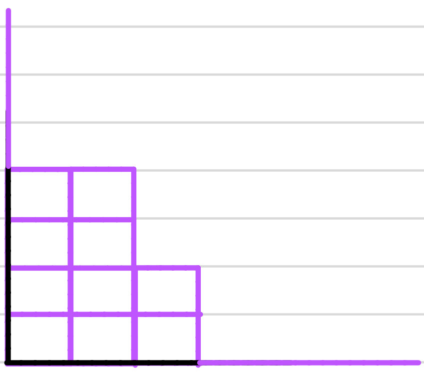
Non-zero images:

$x^2y^2 \mapsto xy^3$

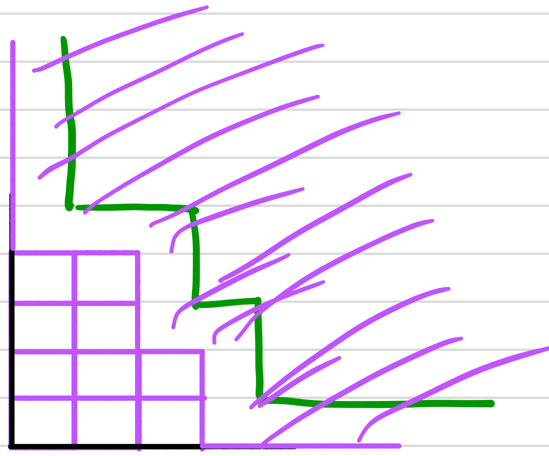


$\tilde{I} + (-1,1) \setminus \tilde{I}$

$I \mapsto 0$



degree $(1,1)$

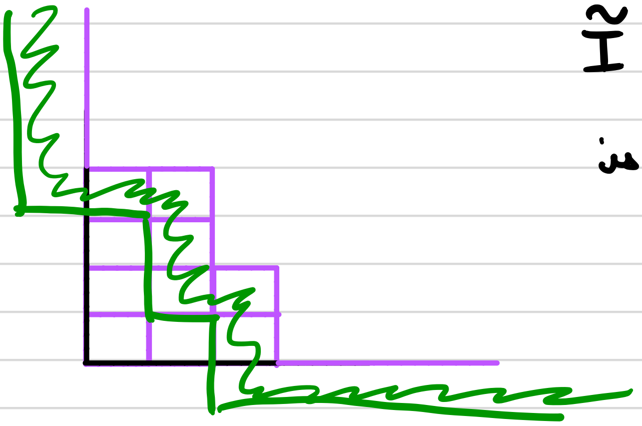


$\tilde{I} + (1,1) \setminus \tilde{I} = \emptyset$

$I \to S/I$

$x^i y^j \longmapsto x^{i+1} y^{j+1} \in I$

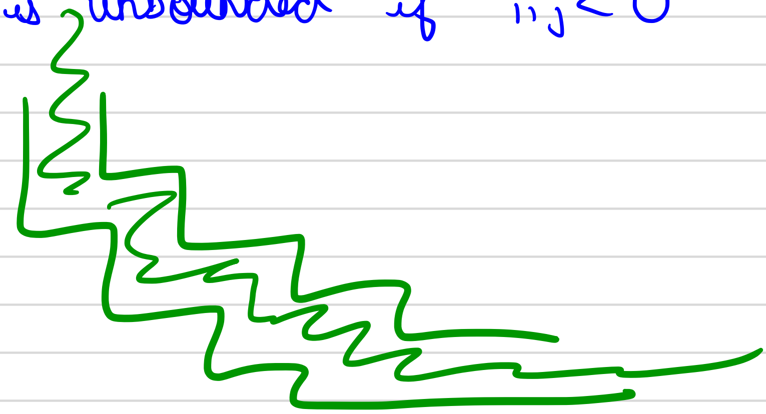
Prop. The set of bounded, connected components of $\tilde{I}+\alpha \setminus \tilde{I}$ forms a basis of $T(I)_\alpha$.



$\tilde{I} + \underline{(-1,-1)} \setminus \tilde{I}$ is unbounded

degree $(-1,-1)$

<u>Cor</u> $\tilde{I}+(i,j) \setminus \tilde{I}$ is unbounded if $i,j<0$



Ex H_2^d is smooth (Based on Haiman's work)

Need to show $\dim T(I) = 2\,\ell(S/I)$.

$$T(I) = \underbrace{\bigoplus_{\substack{i\geq 0\\ j<0}} T(I)_{i,j}}_{T_{pn}(I)} \oplus \underbrace{\bigoplus_{\substack{i<0\\ j\geq 0}} T(I)_{i,j}}_{T_{np}(I)} \oplus \underbrace{\cancel{\bigoplus_{\substack{i\geq 0\\ j\geq 0}} T(I)_{i,j}}}_{T_{pp}(I)} \oplus \underbrace{\cancel{\bigoplus_{\substack{i<0\\ j<0}} T(I)_{i,j}}}_{T_{nn}(I)}$$

$$\dim T_{pn}(I) = \ell(S/I) = \dim T_{np}(I)$$

$\Rightarrow$ H_2^d smooth.

• $I \subseteq K[x,y,z]$ **monomial**

$$T(I) = \bigoplus_{\alpha \in \mathbb{Z}^3} T(I)_\alpha =$$

$$T_{ppn}(I) \oplus T_{pnp}(I) \oplus T_{npp}(I) \oplus T_{nnp}(I) \oplus T_{npn}(I) \oplus T_{pnn}(I) \cancel{\oplus T_{ppp}(I)}^{0} \cancel{\oplus T_{nnn}(I)}^{0}$$

$$T_{ppn}(I) = \bigoplus_{\substack{i \geq 0 \\ j \geq 0 \\ l < 0}} T_{ijl}(I) \qquad T_{nnp}(I) = \bigoplus_{\substack{i < 0 \\ j < 0 \\ l \geq 0}} T_{ijl}$$

Thm [R, S] For any monomial point $I \in H_3^d$ we have

$$\dim_K T_{ppn}(I) = \dim_K T_{nnp}(I) + d$$

$$\dim_K T_{pnp}(I) = \dim_K T_{npn}(I) + d$$

$$\dim_K T_{npp}(I) = \dim_K T_{pnn}(I) + d$$

Cor I is a smooth pt $\iff$, $T_\sigma(I) = 0$ for all $\sigma \in \{nnp, npn, pnn\}$

Pf $\dim T(I) = \underbrace{3d}_{3d} + 2 \sum_{\sigma \in nnp, npn, pnn} \dim T_\sigma(I)$

Cor I is a smooth pt $\iff$ $T_\sigma(I) = 0$
for all $\sigma \in \{nnp, npn, pnn\}$

Ex $(x^2, xy, y^3, xz^2, yz^2, z^4) \in H_3^{10}$

$xy \mapsto z^3$, others to 0

$(-1,-1,3)$

Cor [Behrend - Fantechi] I monomial
$\dim T(I) \equiv d \bmod 2$ for all $I \in H_3^d$

Pf Follows from duality

Goal: $\dim T(I) \leq \dim T(m^n)$

- $\dim T_\sigma(I) \leq \dim T_\sigma(m^n)$ $\quad \sigma \in \{ppn, pnp, npp, npn, nnp, pnn\}$

- $\dim T_\sigma(I) \leq \dim T_\sigma(m^n)$ $\quad \sigma \in \{ppn, pnp, npp\}$

  Assume I is Borel-fixed

Thm [R, S] For I Borel-fixed

$$\dim T_\sigma(I) \leq \dim T_\sigma(m^n) \text{ for } \sigma \in \{ppn, pnp\}$$

with equality in either case iff $I = m^n$.