Localization for relative obstruction theories and stable log maps.

Plan:

I Classical localization, equivariant Chow groups, and generalized Segre classes

II The fixed stack of a torus action

III Virtual classes + main theorem

IV Application to log stable maps

$T = \mathbb{G}_m^r$

1950's : Bott Residue Thm $= \mathbb{G}_m$

(algeom version)

X smooth projective variety w/ T-action

E vector bundle w/ lift of T-action

P a polynomial in chern classes of E $\qquad P = c_1^2 \cdot c_7$

w/ $\deg(P) = \dim X$

then $\int_X P = \int_{X^T} \boxed{}$ ← explicit formula depending on

$$E\big|_{X^T} = \bigoplus_{i \in \mathbb{Z}} E_i$$

also $TX\big|_{X^T} = \bigoplus TX_i$

X^T is smooth $\qquad$ and $T_{X^T} = TX_0 \quad (= TX^{f})$

$N_{X^T} = TX^{m}$

$\boxed{}$ depends on $c(E_i)$ and $c(N_{X^T/X\,i})$

Can understand/prove in terms of

equivariant cohomology (equivariant Chow groups)

$T = \mathbb{G}_m$ acts on any scheme X

Edidin-Graham construct $A^T_*(X)$

formally similar to $A_*(X)$

i.e. functoriality (for T-equivariant morphisms)

Chern classes (for T-eq bundles)

Intersection product for smooth X $\quad (A^*_T(X))$

Specialization map $A_*^T(X) \longrightarrow A_*(X)$

<u>Novelty</u>: $A^*_{\mathbb{G}_m}(pt) = \mathbb{Q}[\lambda]$ $\qquad A^T_{-i}(pt) = \mathbb{Q}\cdot\lambda^i$

$\Rightarrow$ for <u>any</u> X w/ T-action

$A_*^T(X)$ has a natural structure as a $\mathbb{Q}[\lambda]$-module

<u>Localization</u> <u>Thm</u>: $i_*: A_*^T(X^T)_\lambda \xrightarrow{\sim} A_*^T(X)_\lambda$

is an isomorphism after inverting λ.

To recover Bott Residue

observe $\quad i^* i_*(\alpha) = e(N_{X^T/X}) \cap \alpha$

in $A_T^*(X^T) = A^*(X^T) \otimes \mathbb{Q}[\lambda]$

Because torus weights on N are all non-zero

$e(N)$ is invertible in $A_T^*(X^T)_\lambda$

$\Rightarrow$ set $\alpha = \dfrac{1}{e(N)}$ then $i^* i_* \alpha = [X^T]$

$\Rightarrow$ $i_* \alpha = [X]$

projection ~~formula~~

$$\int_{[X]} P = P \cap [X] = P \cap i_*(\alpha) = i^* P \cap \alpha$$

i^*P is a polynomial in (equiv) chern classes of i^*E.

<u>Conclusion</u> $\displaystyle \int_X P(E) = \int_{X^T} \frac{P(i^*E)}{e(N)} \quad \Big| \quad A_*^T(X^T) \cong A_*(X^T) \otimes \mathbb{Q}[\lambda]$

__Key pt__: If we can identify α s.t. $i_*(\alpha) = [X]$ then can reduce integrals over X to integrals over X^T

__Equivariant__ __Residue__: $i_*^{-1} : A^T_*(X) \longrightarrow A^T_*(X^T)_\lambda$

Res:

Related to Segre classes

By def to normal cone

$$\begin{tikzcd} X^T \arrow[r, hook] \arrow[d, squiggly] & X \\ X^T \arrow[r, hook] & C_{X^T/X} \end{tikzcd}$$

can factor

$$\begin{tikzcd} \mathrm{Res} : A^T_*(X) \arrow[rr] \arrow[dr, "\sigma"'] & & A^T_*(X^T) \\ & A^T_*(C_{X^T/X}) \arrow[ur, "\mathrm{Res}"'] & \end{tikzcd}$$

if have embedding $C \hookrightarrow V$

then $\mathrm{Res}([C]) = e(V)^{-1} \cap s_V^*([C])$

(if take $T = \mathbb{G}_m$ w/ standard scaling action, then

$\mathrm{Res}([C]) = \sum s_i(C) \lambda^{d-i}$ $\quad$ $s_i(C)$ = Segre classes of C.

This construction makes sense also for cone __stacks__ which admit T-embeddings in T-vector bundle stacks via same ~~formula~~

define $s(C) = e(V)^{-1} \cap s_V^*(i_*[C])$

$E_0 \to E_1$

define $S(C) = e(V)' \cap s_V^{!}(i_*[C])$

$E_0 \to E_1$

$[E_1/E_0]$

II Fixed stacks

Fixed scheme $\quad T \circlearrowright X \qquad X^T = \bigcap_{t \in T} X^t$

X^t = locus where $t = id$

What is functor of pts?

$S \longrightarrow X^T \iff T$-equiv map $S \xrightarrow{} X$ (trivial $\circlearrowright$ on S)

Def (Romagny) if X is a stack w/ T-action then define X^T to be the stack with

$X^T(S)$ = T-eq maps $S \longrightarrow X$

T-eq morphism
$$\begin{tikzcd} T \times S \arrow[r] \arrow[d, "\pi_1"'] & T \times X \arrow[d] \\ S \arrow[r] \arrow[ur, phantom, "\Rightarrow"] & X \end{tikzcd}$$
($\Rightarrow$ needs to satisfy properties...)

note: natural map $X^T \longrightarrow X$ is not a monomorphism (for T a torus & X DM stack X^T is closed substack)

Ex: $\mathcal{M}_g$ = stack of nodal curves

$\mathcal{M}_g(S)$ = flat families of nodal curves $/S$

$T \circlearrowright \mathcal{M}$ trivially

then $\mathcal{M}_g^T(S)$ = flat families of nodal curves w/ a fiberwise T-action

Ex: $X = [M/G]$ w/ T-action

induced by: (1) an extension of groups

$$1 \to G \to \hat{G} \to T \to 1$$

(2) and action of $\hat{G}$ on M

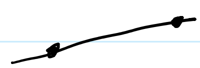
$$X^T = \coprod_{\substack{\varphi : T \to \hat{G} \\ \text{conj classes} \\ \text{of splittings}}} [M^{\varphi(T)} / C(\varphi) \cap G]$$

Subexample: If T acts trivially on $BG = [pt/G]$

then $BG^T = \coprod_{\substack{\text{conj classes} \\ \text{of morphisms} \\ T \to G}} B(C(\varphi))$

Take $G = \mathbb{P}GL_2$ then $BG \subset \mathcal{M}_0$

and see that have copy of $B\mathbb{G}_m \subset BG^T$



Warning: X^T need not be an algebraic stack.

(already fails for $\mathcal{M}_1$)

III Main Thm

Setup: M a DM-stack

$\mathcal{Y}$ an Artin stack

$M \xrightarrow{\pi} \mathcal{Y}$ a T-equiv morphisms

$\mathbb{E} \to L_{M/\mathcal{Y}}$ a perfect obstruction theory

$(\mathbb{E} = [E_{-1} \to E_0]$ $\qquad C_{M/\mathcal{Y}} \hookrightarrow \mathbb{E}$

Manolache $\rightsquigarrow$ this gives rise to a pull-back

$$A_*^T(\mathcal{Y}) \longrightarrow A_*^T(M)$$
$$\pi_{\mathbb{E}}^! \qquad\qquad [M]^{vir} = \pi_{\mathbb{E}}^!([\mathcal{Y}])$$

When $\mathcal{Y} = pt$ then virtual localization thm

M^T has p.o.T. given by $\mathbb{E}^f$

$$i_*\left(e(\mathbb{E}^{vm})^{-1} \cap [M^T]^{vir}\right) = [M]^{vir}$$

In relative case $\qquad M^T \xrightarrow{\pi^T} \mathcal{Y}^T$ has rel. p.o.T given by $\mathbb{E}^f$

get natural class $\alpha \in \widehat{A}_*^T(\mathcal{Y}^T)_\lambda$

by taking $S(C_{\mathcal{Y}^T/\mathcal{Y}})$

$$\boxed{[M]^{vir} = i_*\left(e(N^{vir})^{-1} \cap \pi^{T\,!}_{\mathbb{E}^f}(\alpha)\right)}$$

(Tech hypotheses :
$$\begin{tikzcd} & M^T \arrow[dl] \arrow[d] \\ U & \mathcal{Y}^T \arrow[l, phantom, "\subset"] \end{tikzcd}$$
algebraic $\qquad$ open

+ existence of resolutions

<u>Log</u> <u>stable</u> <u>maps</u> : $\quad X$ projective log smooth scheme

get moduli space $M = \overline{M}_{g,n}(X, \beta)$ of log stable maps.

If S is a log scheme then

$M(S) =$ log smooth curve $\begin{array}{c} C \xrightarrow{f} X \\ \downarrow \pi \\ S \end{array}$ f log morphism

represented by a DM-stack w/ log structure

M comes w/ P.O.T. rel to singular base $\mathcal{Y}$

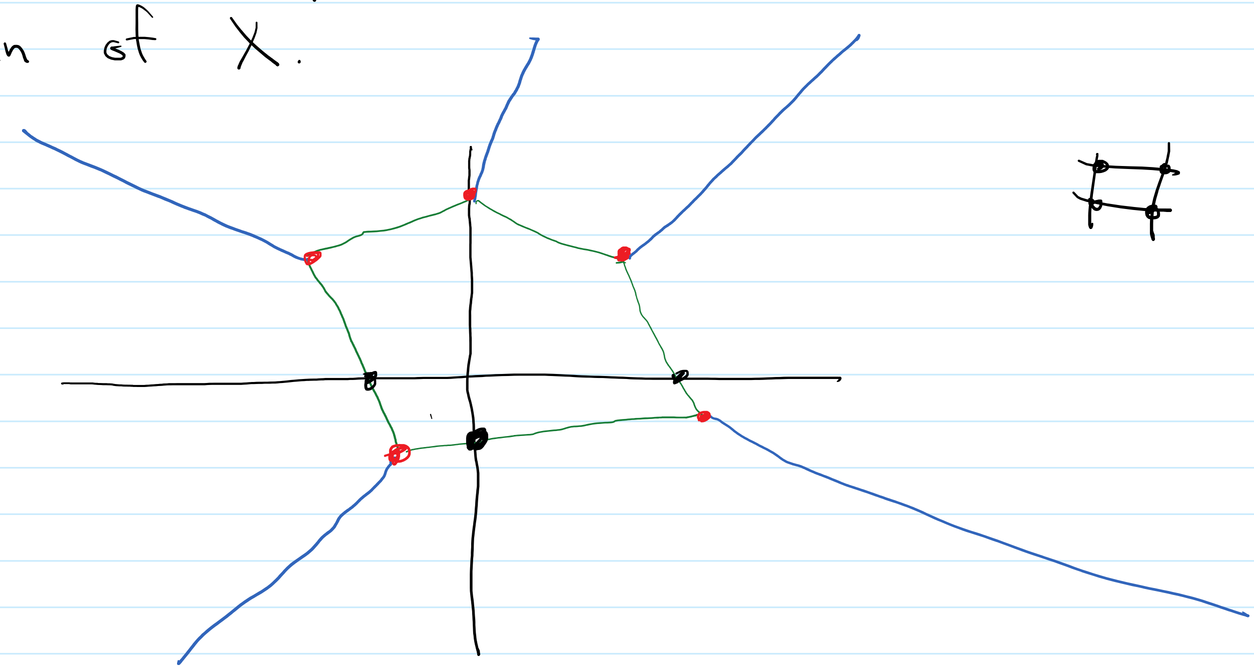
$\mathcal{Y}$ etale locally of form $[U/T_U]$ $U =$ affine toric variety, $T_U =$ its torus

Near a pt of $\mathcal{Y}^T \longrightarrow \mathcal{Y}$ given by $T \to T_U$

$U^T \subset U$

$$\boxed{\text{General question: Given toric variety } V \text{ and a subtorus } T \subset T_V \text{ can we compute } \operatorname{Res}([\mathcal{U}]) \in A^T_*(V^T)_x\,?}$$

For $M = \overline{M}^{\log}_{g,n}(X, \beta)$ relevant polyhedral complex is moduli space of tropical curves in fan of X.



On fixed locus, corresponding trop curve has special features $\Rightarrow$ explicit description of $\mathcal{Y}^T \longrightarrow \mathcal{Y}$

In principle (but not in reality) can compute class $\alpha \in \hat{A}^T_*(\mathcal{Y}^T)$
Know in advance that α can be written in terms of closed strata of $\mathcal{Y}^T$ together with standard cohomology classes.

Last step: Given closed stratum $Z \subset \mathcal{Y}^T$, want explicit description of $\pi_{LE}^{T!}([Z])$ i.e. virtual fund class of the stratum.

<u>Two</u> <u>stages</u>

I Decompose into connected components of contracted locus.

II Locus in $\overline{M}_{g,n}$ corresponding to a tropical graph $\rightsquigarrow$ Intersection of DR-type cycle with ∂-stratum.