q-bic Hypersurfaces

Throughout. $\Bbbk :=$ algebraically closed field of characteristic $p > 0$

$q := p^e$ for some $e \in \mathbb{Z}_{>0}$.

Defn A q-bic hypersurface in $\mathbb{P}^{n+1}_{\Bbbk}$, or q-bic n-fold, is a hypersurface of degree $q+1$ given by an equation of the form

$$f := \sum_{i,j=0}^{n+1} a_{ij} x_i x_j^q \in \Gamma(\mathcal{O}_{\mathbb{P}^{n+1}}(q+1)).$$

Examples.

① Fermat hypersurfaces. $(x_0^{q+1} + x_1^{q+1} + \cdots + x_{n+1}^{q+1} = 0) \subset \mathbb{P}^{n+1}$

② Hermitian curve. $(x_0^q x_1 - x_0 x_1^q + x_2^{q+1} = 0) \subset \mathbb{P}^2$

③ q-cuspidal curve. $(x_0 x_1^q + x_2^{q+1} = 0) \subset \mathbb{P}^2$

Classification of q-bic hypersurfaces.

0. Equations live in distinguished linear subspace:

$$\underbrace{\Gamma(\mathcal{O}_{\mathbb{P}^{n+1}}(1))}_{:= V} \otimes \underbrace{\Gamma(\mathcal{O}_{\mathbb{P}^{n+1}}(1))^{(q)}}_{:= V^{(q)}} \underset{\text{mult}}{\subset} \underbrace{\Gamma(\mathcal{O}_{\mathbb{P}^{n+1}}(q+1))}_{= \mathrm{Sym}^{q+1}(V)}$$

$\Gamma(\mathcal{O}(q))$
$\cup$
$\Gamma(\mathcal{O}(1))^{(q)}$
linear subsp. spanned by q-powers: $x^q + y^q = (x+y)^q$

1. Parameter space of q-bics given by

$$\mathbb{P}(V \otimes V^{(q)})$$

$\rightsquigarrow$ points of $\mathbb{P}(V \otimes V^{(q)})$ $\leftrightarrow$ $V^\vee \otimes V^{(q),\vee} \to k$ linear $\leftrightarrow$ bilinear forms b/t $V^\vee$ & $V^{(q),\vee}$

This suggests the following convenient way of labelling points of the parameter space:

$$f = \sum_{i,j=0}^{n+1} a_{ij} x_i x_j^q \longleftrightarrow A := (a_{ij})_{i,j=0}^{n+1}$$

Gram matrix of f

2. $V \otimes V^{(q)} \subset \mathrm{Sym}^{q+1}(V)$ stable under the natural $GL(V)$-action by coordinate substitutions, so form the moduli stack:

$$\underline{q\text{-bic}}^n := \left[\mathbb{P}(V \otimes V^{(q)}) / PGL(V) \right]$$

Classification Theorem. [Hefez '85, Beauville '90, Hoai Hoang '16, –'18, Kadyrsizova–Kenkel–Page–Singh–Smith–Vraciu–Witt '20]

The stack $\underline{q\text{-bic}}^n$ is of dimension 0 and its points are in bijection with partitions of size $\leq n+1$, the correspondence being

$$\lambda = (\lambda_1 \geq \cdots \geq \lambda_m) \longleftrightarrow \text{orbit of } A_\lambda := J_{\lambda_1} \oplus \cdots \oplus J_{\lambda_m} \oplus (1)^{\oplus n+1-|\lambda|}$$

where $J_\mu := \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix} \in \mathrm{Mat}_{\mu\times\mu}$.

Examples.

1. The Fermat q-bic hypersurface

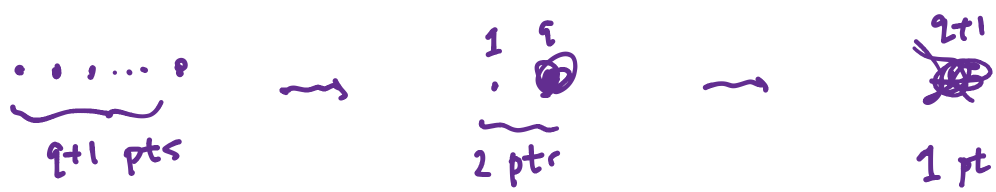
$$(x_0^{q+1} + x_1^{q+1} + \cdots + x_{n+1}^{q+1} = 0) \subset \mathbb{P}^{n+1}$$

is the unique smooth q-bic, up to projective equivalence.

2. q-bic 0-folds in $\mathbb{P}^1$ look like:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

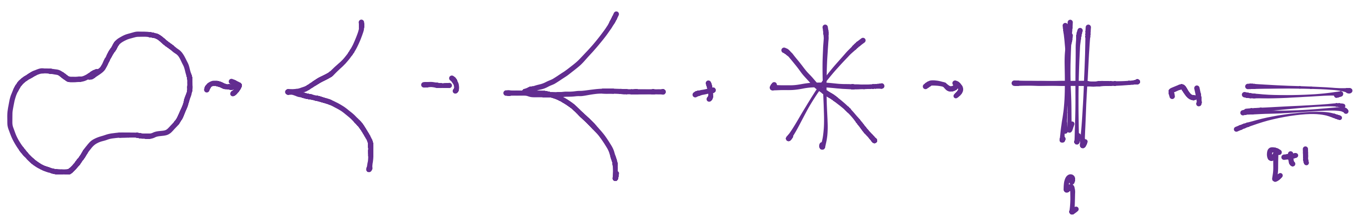
$$x_0^{q+1} + x_1^{q+1} \qquad x_0 x_1^q \qquad x_0^{q+1}$$



3. q-bic curves in $\mathbb{P}^2$ look like:

$$\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \rightsquigarrow \left(\begin{array}{cc|c} 0 & 1 & \\ & 0 & \\ \hline & & 1 \end{array}\right) \rightsquigarrow \begin{pmatrix} 0 & 1 & \\ & 0 & 1 \\ & & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix} \rightsquigarrow \left(\begin{array}{cc|c} 0 & 1 & \\ & 0 & \\ \hline & & 0 \end{array}\right) \rightsquigarrow \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix}$$

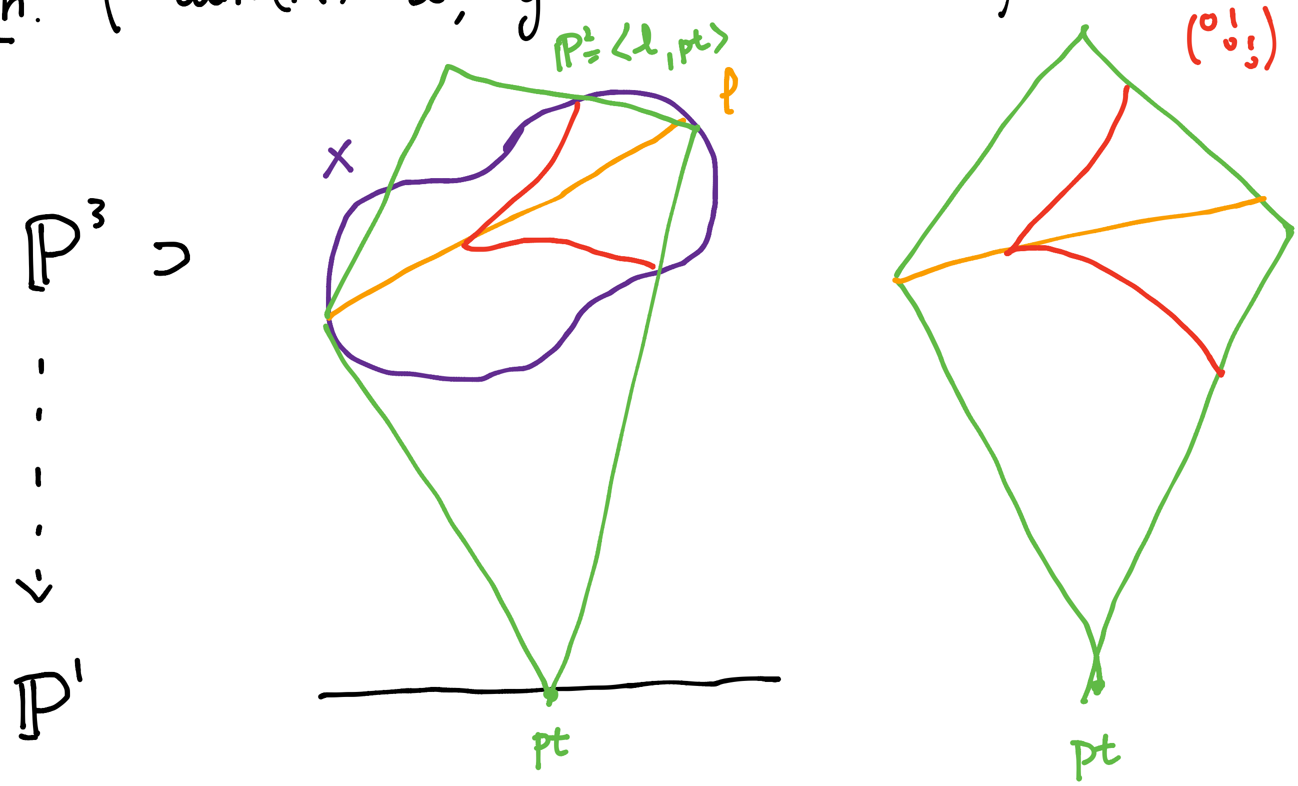
$$x_0^{q+1} + x_1^{q+1} + x_2^{q+1} \qquad x_0 x_1^q + x_2^{q+1} \qquad x_0 x_1^q + x_1 x_2^q \qquad x_0^{q+1} + x_1^{q+1} \qquad x_0 x_1^q \qquad x_0^{q+1}$$



Old Observation. [Shioda '74]

> A smooth q-bic hypersurface of dim ≥ 2 is purely inseparably unirational.

Sketch. ($\dim(X) = 2$, general case similar)



$$\begin{tikzcd}
X & \tilde{X} \arrow[l] \arrow[d] & \tilde{X}' \arrow[l] \arrow[d] \arrow[dl, phantom, "\lrcorner"] & \mathbb{P}^1 \times \{ \arrow[l, dashed] & \mathbb{P}^1 \times \mathbb{P}^1 \arrow[l, "\sim"'] \\
& \mathbb{P}^1 & \mathbb{P}^1 \arrow[l] & &
\end{tikzcd}$$

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Linear Geometry of q-bics.

$X \hookrightarrow \mathbb{P}^{n+1}$ smooth q-bic n-fold

$F_r(X) :=$ Fano scheme of $\mathbb{P}^r \subset X$ $\hookrightarrow \mathrm{Gr}(r+1, n+2)$.

Prop. [–] $F_r(X)$ is smooth of dimension $(r+1)(n+2-2(r+1))$.

First Case. $\left.\begin{array}{l} n=2 \\ r=1 \end{array}\right\}$ $X \hookrightarrow \mathbb{P}^3$ q-bic surface

$F_1(X) \hookrightarrow \mathrm{Gr}(2,4)$ dim $= 0$

Second Case. $\left.\begin{array}{l} n=3 \\ r=1 \end{array}\right\}$ $X \hookrightarrow \mathbb{P}^4$ q-bic threefold

$\underbrace{F_1(X)}_{:= S} \hookrightarrow \mathrm{Gr}(2,5)$ dim $= 2$

Theorem. [–]

1. S smooth projective surface of general type.
   In fact: $\omega_S \cong \mathcal{O}_S(2q-3)$.

2. If $q > 2$, S does not lift to $W_2(k)$

3. $$\mathrm{Alb}(S) \xrightarrow{\text{isogenous}} \mathrm{Jac}^3(X)$$

   and both are further isogenous to

   $$\prod^{q^2+1} \mathrm{Jac}(C) \qquad C := \text{smooth } q\text{-bic curve}.$$

4. S is supersingular and satisfies the Tate conjecture.

5. $$\frac{1}{2} b_1(S) = (q^2+1)\binom{q}{2} = h^1(\mathcal{O}_S).$$

   if $q = p$ prime

Indication on Proofs.

① & ②

$$T_S \cong \mathcal{H}om(\mathcal{I}, \mathcal{I}^{\perp}/\mathcal{I}) \cong \mathcal{I}^{\vee} \otimes \mathcal{O}_S(1-q) \cong \mathcal{I} \otimes \mathcal{O}_S(2-q)$$

$$\rightsquigarrow \quad \Omega_S \cong \mathcal{I}^{\vee} \otimes \mathcal{O}_S(q-2).$$

$$\begin{array}{l} \Omega_S \\ \cup \\ \mathcal{O}_S(q-2) \text{ ample} \to \text{no lift by Langer} \end{array}$$

③ & ④. Look at tautological incidence correspondence

$$\begin{tikzcd}
\tilde{S} \arrow[dd, "\text{blow up}"'] \arrow[dr, phantom, "\subset"] & & \tilde{Z} \arrow[ll, "\text{finite pure insep}"'] \arrow[dl] \arrow[dr, "\text{blow up}"] & \\
 & I \arrow[dl] \arrow[dr] & & Z \arrow[dl, phantom, "\supset"] \\
S & & X &
\end{tikzcd}$$

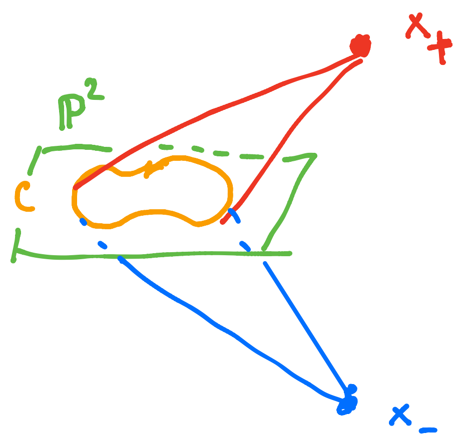
Then study explicit set of cycles + use the big automorphism group acting on everything.

⑤ Betti numbers via Deligne–Lusztig theory for the finite unitary group $SU_3(q)$.

Coherent cohomology via degeneration :

$$\begin{tikzcd} X \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \subset & \mathcal{X} \arrow[d] & \supset & X_0 \arrow[d] \arrow[dl, phantom, "\urcorner", very near start] \\ 1 & \in & \mathbb{A}^1 & \ni & 0 \end{tikzcd}$$

$\leftarrow$ type $\left(\begin{array}{cc|ccc} 0 & 1 & & & \\ & 0 & & & \\ \hline & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{array}\right)$



$$T \cong \mathbb{P}(T_C(-1) \oplus \mathcal{O}_C)$$

$$\begin{tikzcd} S \arrow[d] & \subset & \mathcal{S} \arrow[d] & \supset & S_0 \arrow[d] \arrow[dr] & & T \arrow[ll, "\nu"'] \arrow[dr] & \\ 1 & \in & \mathbb{A}^1 & \ni & 0 & C & & C \arrow[ll, "Fr^2"'] \end{tikzcd}$$

$$0 \longrightarrow \mathcal{O}_{S_0} \longrightarrow \nu_* \mathcal{O}_T \longrightarrow \mathcal{T} \longrightarrow 0$$

Localizing to formal neighbourhood of singularity, reduce to a computation on C, upon which

$$\mathcal{T} \cong \bigoplus_{d=0}^{2p^2-p-2} \mathcal{T}_d \quad \text{as } \mathcal{O}_C\text{-modules.}$$

$\longrightarrow$ Main task: compute $h^0(\mathcal{T}_i)$.

Compute cohomology by degrees:

$p=2$

1	3
0	1
0	

$p=3$

3	6	7
1	3	6
0	1	3
0	0	1

$p=5$

10	15	18	19	18
6	10	15	18	19
3	6	10	15	18
1	3	6	10	15
0	1	3	6	10
0	0	1	3	6
0	0	0	1	3
0	0	0	0	1
0	0	0	0	

$p=7$

21	28	33	36	37	36	33
15	21	28	33	36	37	36
10	15	21	28	33	36	37
6	10	15	21	28	33	36
3	6	10	15	21	28	33
1	3	6	10	15	21	28
0	1	3	6	10	15	21
0	0	1	3	6	10	15
0	0	0	1	3	6	10
0	0	0	0	1	3	6
0	0	0	0	0	1	3
0	0	0	0	0	0	1
0	0	0	0	0	0	

Conclusion. $h^0(\mathcal{T}) = (p^2+1)\binom{p}{2} + \binom{p}{3}$.

$$= h^1(\mathcal{O}_{S_0})$$

Now have the following inequalities:

$$(p^2+1)\binom{p}{2} = \tfrac{1}{2} b_1(S) \le h^1(\mathcal{O}_S) \le h^1(\mathcal{O}_{S_0}) = (p^2+1)\binom{p}{2} + \binom{p}{3}$$

theory of Picard schemes

uppersemicty.

<u>Final Step</u>. Show that $\binom{p}{3}$ many H^1-classes do not lift along degeneration.

$$\begin{tikzcd} S \arrow[dr, "\varphi"] \arrow[dd] & \\ & C \times \mathbb{A}^1 \arrow[dl] \\ \mathbb{A}^1 & \end{tikzcd} \quad \rightsquigarrow \quad R^1\varphi_* \mathcal{O}_S \text{ coherent } \mathcal{O}_{C\times\mathbb{A}^1}\text{-module}$$

which is $\mathbb{G}$-equivariant, action on $\mathbb{A}^1$ of wt q^2-1

Thus by general theory of $\mathbb{G}_m$-equivariant sheaves on $\mathbb{A}^1$, obtain the following:

$$R^1\varphi_* \mathcal{O}_S \cong \bigoplus_{d=0}^{p^2-2} \mathcal{H}_d$$

$\leftarrow$ coherent $\mathcal{O}_C$-modules fitting into SES

$$0 \to \mathcal{T}_d \to \mathcal{H}_d \to \mathcal{T}_{p^2+d-1} \to 0.$$

Examine the sequences

$$0 \to \mathcal{T}_{ip} \to \mathcal{H}_{ip} \to \mathcal{T}_{p^2+ip-1} \to 0$$

$$\mathcal{T}_{p^2+ip-1} \cong \mathcal{O}_S \otimes \mathrm{Sym}^{p-2-i}(V)$$

$\dim(V) = 3$

for $i = 1, \dots, p-2$.

$\rightsquigarrow$ These sequences do not split at $1 \in \mathbb{A}^1$, so the classes from $\mathcal{T}_{p^2+ip-1}$ do not lift. There are

$$\sum_{i=1}^{p-2} \dim\left(\mathrm{Sym}^{p-2-i}(V)\right) = \sum_{i=1}^{p-2} \binom{(p-2-i)+2+i}{i} = \binom{p}{3}$$

many. — //

Looking Ahead.

1. Other interesting structures and special varieties?

2. Varieties defined similarly?

3. q- polynomiality of cohomology, etc?

— THANK YOU! —