Tropicalization and logarithmic ~~compactification~~ bordification of reductive groups

Classical tropicalization

Setup:

$K := \mathbb{C}((t^{\mathbb{R}}))$ = field of Hahn series

$\rightsquigarrow \mathrm{val}: K \twoheadrightarrow \overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$

$\rightsquigarrow K = \overline{K}$

$T \cong \mathbb{G}_m^n$ an algebraic torus

$N = \mathrm{Hom}(\mathbb{G}_m, T)$

$M = \mathrm{Hom}(T, \mathbb{G}_m) = \mathrm{Hom}(N, \mathbb{Z})$

$\hookrightarrow$ Tropicalization map

$$\begin{tikzcd} T^{an} \approx T(K) \arrow[r, "\mathrm{trop}"] \arrow[d, phantom, "\cong"] & N_{\mathbb{R}} \arrow[d, phantom, "\cong"] \\ (K^*)^n & \mathbb{R}^n \\ x \arrow[r, mapsto] & (\mathrm{val}(x_1), \dots, \mathrm{val}(x_n)) \end{tikzcd}$$

Berkovich analytic space

Usual next step: Study subvarieties $Y \subseteq T$

via their tropicalization

$$\mathrm{trop}(Y(K)) \subseteq N_{\mathbb{R}}$$

rat. pol. fan [Bieri - Groves '84]

Question: What if we replace T by a reductive group G?

For simplicity:

	reductive	semisimple	semisimple of adjoint type
G	GL_n	SL_n	$PGL_n = GL_n/\mathbb{G}_m$

Def.: (Goldman - Iwahori space)

Let $K = (K, |\cdot|)$ be a non-Archimedean field and V = fin.-dim. K-vector space. A non-Archimedean norm on V is a map $\|.\|: V \longrightarrow \mathbb{R}_{\geq 0}$ s.t.

(i) $\|v\| = 0 \iff v = 0$

(ii) $\|\lambda \cdot v\| = |\lambda| \cdot \|v\| \quad \forall \lambda \in K \;\&\; v \in V$

(iii) $\|v + w\| \leq \max\{\|v\|, \|w\|\} \quad \forall v, w \in V$

The space $\mathcal{N}(V) = \{\text{non-Arch. norms on } V\}$ is called Goldman-Iwahori space of V.

Topology on $\mathcal{N}(V)$: take the weakest topology s.t.

$$\begin{aligned} ev_v : \mathcal{N}(V) &\longrightarrow \mathbb{R}_{\geq 0} \\ \|.\| &\longmapsto \|v\| \end{aligned}$$

is continuous.

Example: Let $e := (e_1, \dots, e_n)$ be an ordered basis of V

and $c := (c_1, \dots, c_n) \in \mathbb{R}^n$. Then

$$\begin{aligned} \|.\|_{e,c} : V &\longrightarrow \mathbb{R}_{\geq 0} \\ x = x_1 e_1 + \dots + x_n e_n &\longmapsto \max_{i=1,\dots,n} \{ |x_i| e^{c_i} \} \end{aligned}$$

is a non-Arch. norm on V.

Prop.: For every non-Arch. norm $\|.\|$ on V there

is a basis e & $c \in \mathbb{R}^n$ s.t.

$$\|.\| = \|.\|_{e,c}$$

In fact we obtain closed immersions

$$\begin{aligned} \mathbb{R}^n &\hookrightarrow \mathcal{N}(V) \\ c &\longmapsto \|.\|_{e,c} \end{aligned}$$

for every ordered basis e of V

Its image is called an apartment of $\mathcal{N}(V)$.

$\hookrightarrow$ Bruhat-Tits buildings

1-1

tori $T \hookrightarrow GL_n$

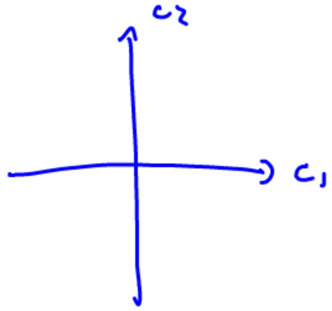
Set $\mathcal{X}(V) := \mathcal{N}(V)/_{\sim}$ where

$$\|.\| \sim \|.\|' \iff \exists c \in \mathbb{R}_{>0} \text{ s.t. } \|v\| = c\|v\|' \quad \forall v \in V$$

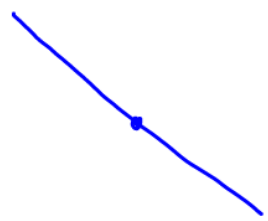
<u>Example:</u> $V = \mathbb{C}^2$ with $|.| : \mathbb{C} \to \mathbb{R}_{\geq 0}$ the trivial norm
$$a \mapsto \begin{cases} 1, & a \neq 0 \\ 0, & \text{else} \end{cases}$$

- One apartment

in $\mathcal{N}(V)$



in $\mathcal{X}(V)$



In $\mathcal{X}(V)$ every element is equivalent to $\|.\|_{(e_1,e_2),(c,1)}$ for some basis (e_1,e_2) of V & $c \in \mathbb{R}$.

<u>Claim:</u> If $c \geq 0$, then $\|.\|_{(e_1,e_2),(c,1)} \sim \|.\|_{(e_1',e_2'),(c,1)}$ for any other basis (e_1',e_2') of V (up to reordering)

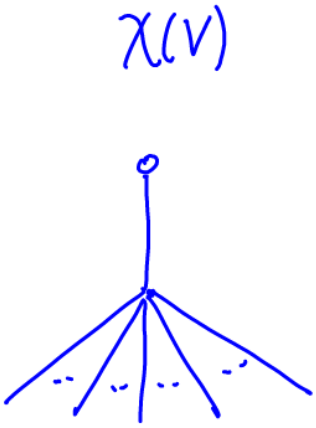
<u>Sketch of proof:</u> Let $c > 0$. Write $v = \lambda_1 e_1 + \lambda_2 e_2 = \lambda_1' e_1' + \lambda_2' e_2'$

$$\Rightarrow \|v\|_{(e_1,e_2),(c,1)} = \max\{|\lambda_1| e^c, |\lambda_2| e^1\} = \begin{cases} e^c & \text{if } \lambda_1 \neq 0 \\ 1 & \text{if } \lambda_1 = 0 \end{cases}$$

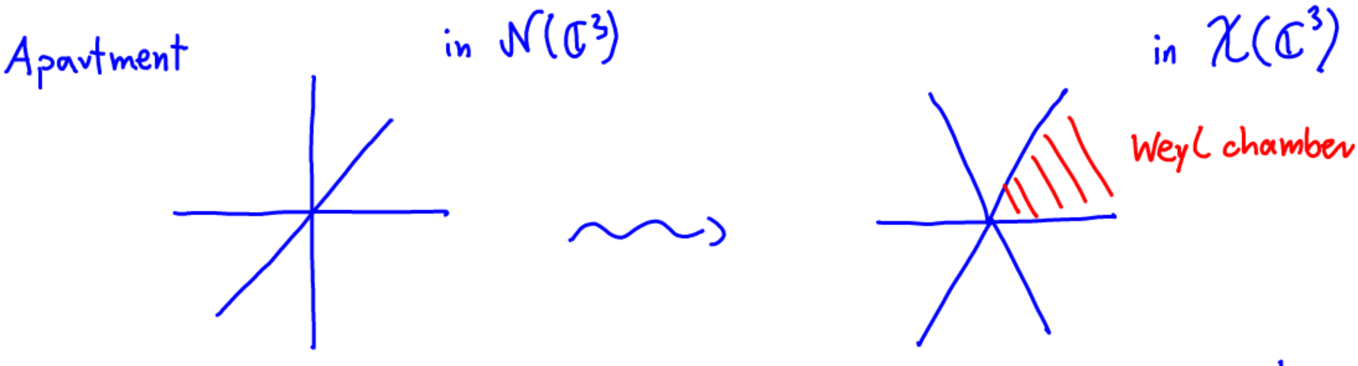
$$\|v\|_{(e_1',e_2'),(c,1)} = \max\{|\lambda_1'| e^c, |\lambda_2'| e^1\} = \begin{cases} e^c & \text{if } \lambda_1' \neq 0 \\ 1 & \text{if } \lambda_1' \neq 0 \end{cases}$$

$\square$

So in $\mathcal{X}(V)$ we need to identify the positive rays of each apartment



Example: $V = \mathbb{C}^3$



- A_2 root system
- fan of Losev-Manin space

More Properties of $\mathcal{N}(V)$

- Let $L|K$ be a non-Archimedean extension. Then the natural map

$$\begin{aligned} \mathcal{N}(V_L) &\longrightarrow \mathcal{N}(V) \\ \|.\| &\longmapsto \|.\|\big|_V : V \to V_L \xrightarrow{\|.\|} \mathbb{R}_{\geq 0} \end{aligned}$$

is a continuous surjection.

- There is a natural $GL_n(K)$-operation

$$\begin{aligned} GL_n(K) \times \mathcal{N}(V) &\longrightarrow \mathcal{N}(V) \\ (g\,,\ \|.\|) &\longmapsto \|.\| \circ g^{-1} \end{aligned}$$

Definition (Tropicalization map)

$$GL_n(K) \approx GL_n^{an} \xrightarrow{\ trop\ } \mathcal{N}(\mathbb{C}^n)$$

$$\ni \quad A \longmapsto \underbrace{\|\cdot\|_{S,\, val(\vec{\lambda})}}_{\in \mathcal{N}(K^n)} \Big|_{\mathbb{C}^n}$$

Let $S = [v_1, \dots, v_n] \in GL_n(K)$

s.t.

$S^{-1} A S$ = Jordan canonical form

$$= \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

For a general reductive group G

↳ G^{trop} = Bruhat-Tits building of $G/Z(G)$ ← semisimple of adjoint type

extended by $N_{\mathbb{R}}$ for all tori

$T \hookrightarrow G$ with $N = \mathrm{Hom}(\mathbb{G}_m, T)$

↳ Integral structure in each apartment coming from $N \subseteq N_{\mathbb{R}}$

↳ $trop_G : G^{an} \longrightarrow G^{trop}$

Thm (Mumford, Brion-Kumar, Martens-Thaddeus, Remy-Thuillier-Werner, U)

(1) For every rat. poly. decomposition Σ of G^{trop} into a (stacky) cone complex there is a separated DM-stack $\overline{G}^{\Sigma}$ that is locally of finite type $/\mathbb{C}$ and an open & dense immersion $G \hookrightarrow \overline{G}^{\Sigma}$ s.t.

(i) $G \hookrightarrow \overline{G}^{\Sigma}$ is a toroidal embedding

(ii) $\overline{G}^{\Sigma}$ is smooth if & only if the cones in Σ are generated a lattice basis

(iii) the conjugation operation $G \times G \to G$ extends to $\overline{G}^{\Sigma}$

(iv) the closure of the locus of GIT-stable points fulfils the valuative criterion of properness

(2) The tropicalization map $trop_G : G^{an} \to G^{trop}$ is G-equivariant and there is a natural section $J : G^{trop} \to G^{an}$ such that $\rho := J \circ trop_G$ is a strong deformation retraction onto a closed subset of G^{an}, its <u>skeleton</u>

$\Rightarrow$ $trop_G$ is continuous & surjective.

Proof: For an apartment $\mathcal{A} \subseteq \mathcal{N}(\mathbb{C}^n)$ set

$$\Sigma_{\mathcal{A}} := \Sigma \cap \mathcal{A}$$

and denote by $T_{\mathcal{A}}$ the corresponding torus $T_{\mathcal{A}} \hookrightarrow G$

Define
$$\overline{G}^{\mathcal{A}} := G \times^{T_{\mathcal{A}}} \underbrace{X(\Sigma_{\mathcal{A}})}_{\substack{T_{\mathcal{A}}\text{- toric variety} \\ \text{defined by } \Sigma_{\mathcal{A}}}}$$

and set
$$\overline{G}^{\Sigma} := \varinjlim_{\substack{\mathcal{A} \text{ apartment} \\ \text{of } \mathcal{N}(\mathbb{C}^n)}} \overline{G}^{\mathcal{A}} \qquad \blacksquare$$

Remarks: (i) The case of non-constant coefficients is work-in-progress with Lorenzo Fantini

↳ affine Bruhat-Tits buildings, toric schemes

(ii) $\overline{G}^{\Sigma}$ is not (!) the wonderful compactification of de Concini - Procesi.

↳ see below.

Wonderful tropicalization

Choose a stacky fan structure Σ^W on the Weyl chamber

$$X^{\Sigma^W} \supseteq PGL_n(K) \approx PGL_n^{an} \xrightarrow{trop^W} W_n = \text{Weyl chamber}$$

Martens-Thaddeus wonderful compactification

$$\| \qquad X^{\Sigma^W}(R)$$

$$\Downarrow$$

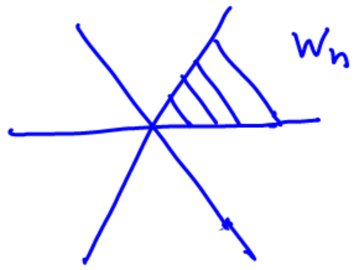
$$\mathcal{X}/R \longmapsto \text{dual tropical curve}$$

semistable degeneration of $\mathbb{P}^1$ to a chain of $\mathbb{P}^1$

$+$

PGL_n-bundle on $[\mathcal{X}/\mathbb{G}_m]$

stable w.r.t. Σ^W



W_n = moduli space of rational tropical chain with unordered points

$= LM_n^{trop}/S_n$

Thm (Remy-Thuillier-Werner, U)

(i) The wonderful tropicalization

$$\mathrm{trop}^W : G^{an} \longrightarrow W(G)$$

has a continuous section $J: W(G) \to G^{an}$ s.t. $J \circ \mathrm{trop}^W$ is a strong deformation retraction onto a non-Arch. skeleton of G^{an}

(ii) The following diagram commutes

$$\begin{tikzcd} & G^{trop} \arrow[d, "/G(K)"] \\ G^{an} \arrow[ur, "\mathrm{trop}"] \arrow[dr, "\mathrm{trop}^W"'] & W(G) \end{tikzcd}$$

$\|\cdot\|_{b,c} \longmapsto \|\cdot\|_{e,c}$ s.t. $c_1 \geq c_2 \geq \cdots \geq c_n$

Standard basis of $\mathbb{C}^n$

Logarithmic compactifications

Recall: $\mathbb{G}_m^{\log} : \mathrm{LogSch} \longrightarrow \mathrm{Groups}$

$$(X, M_X) \longmapsto M_X^{gp}(X)$$

$\hookrightarrow$ $0 \to \mathbb{G}_m \longrightarrow \mathbb{G}_m^{\log} \longrightarrow \mathbb{G}_m^{trop} \longrightarrow 0$

$$\mathbb{G}_m^{trop}(X, M_X) = M_X^{gp}(X) / \mathcal{O}_X^*(X)$$

$\hookrightarrow$ $\mathbb{G}_m^{\log}$ is :
- a group
- proper
- log-smooth

but not a scheme with a log str. !

$\hookrightarrow$ $(\mathbb{G}_m^{\log})^n$ is the "minimal proper $\mathbb{G}_m^n$-toric variety"

fan of $(\mathbb{G}_m^{\log})^n$

$$\shortparallel$$

$(\mathbb{G}_m^{trop})^n \approx (\mathbb{R}^n, +)$.

without polyhedral structure

Def.: Let G be a reductive group & $T \hookrightarrow G$ an algebraic torus. Set

$$G^{\log,T} := G \times^T T^{\log}$$

$T = N \otimes \mathbb{G}_m \rightsquigarrow T^{\log} = N \otimes \mathbb{G}_m^{\log}$

The logarithmic compactification of G is defined to be the functor

$$G^{\log} := \bigcup_{\substack{T \hookrightarrow G \\ \text{torus}}} G^{\log,T}$$

$\hookrightarrow$ natural open immersion $G \hookrightarrow G^{\log}$.

Its tropicalization is defined to be the functor

$$G^{\text{trop}} := \bigcup_{\substack{T \hookrightarrow G \\ \text{torus}}} G^{\text{trop},T} \rightsquigarrow T^{\text{trop}} = N \otimes \mathbb{G}_m^{\text{trop}}.$$

Obs.: $G \backslash G^{\log} \simeq G^{\text{trop}}$

Thm.: (U) Let G be a reductive group

(i) The functor $G^{\log}$ is universally closed, separated and locally of finite type. It admits a log-étale cover by all $\overline{G}^{\Sigma}$ for a polyhedral decompositions Σ of G^{trop}

(ii) The operation of G on G by conjugation extends to an operation on $G^{\log}$ and on G^{trop} s.t. the natural smooth, strict and surjective tropicalization morphism

$$G^{\log} \longrightarrow G^{trop}$$

is G-equivariant.

(iii) The tropicalization map

$$GL_n^{\log} \longrightarrow GL_n^{trop}$$

is a universal framed toric vector bundle

$$\begin{tikzcd} & E \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & GL_n^{\log} \arrow[d] \\ T \arrow[ur, bend left] & X(\Delta) \arrow[r] & GL_n^{trop} \end{tikzcd}$$

↳ Generalizes to parabolic bundles using root stacks along logarithmic boundaries.