Iterated p-adic integration

p-adic integration

joint work with Daniel Litt

Let X be a proper algebraic curve defined over a finite extension $\mathbb{K}$ of $\mathbb{Q}_p$.

Q Can I integrate 1-forms on X?

A regular 1-form $\omega \in \Omega^1(X)$ makes sense

Does a path γ in X makes sense? Can we define $\int_\gamma \omega$?

Idea: Hunt for a primitive function F_ω with $dF_\omega = \omega$, so $\int_\gamma \omega = F_\omega(\gamma(1)) - F_\omega(\gamma(0))$.

Given a point $x \in X$, we can define the "residue disc" around x. These are all points a "distance less than 1 around x." Now, $B(x,1) \cong \{t \in \mathbb{C}_p : \|t\|_p < 1\}$

The 1-form can be expressed as a power series

$$\omega = \sum_{n=1}^{\infty} a_n t^n \, dt.$$

We can integrate it term-by-term

$$\int \omega = \sum_{n=1}^{\infty} a_n \frac{t^{n+1}}{n+1}.$$

Weird Fact: When we integrate the radius of convergence gets smaller. Since $\|p^{-1}\|_p = p > 1$,

$$\left\|\frac{a_n}{n+1}\right\|_p \geq \left\|\frac{a_n}{n+1}\right\|_p \text{ with } > \text{ when } p \mid n+1.$$

This causes all sorts of weird technical issues.

Huge Problem: The topology on X is totally disconnected. We can integrate ω on the disc, but we cannot analytically continue the function.

We'll now give two approaches to p-adic integration.

Abelian Integration

Given a base-point $p \in X$, we can define the Abel-Jacobi map algebraically,

$$i_p : X \longrightarrow J(X).$$

__Amazing Fact__ Over p-adic fields, $\mathcal{L}og$ on Abelian varieties is globally defined,

$$\mathcal{L}og : J(X) \longrightarrow \mathcal{L}ie\, J.$$

Why? By power series arguments, it exists in a neighborhood $\mathcal{U}$ of e. Since $J(\overline{\mathbb{F}}_p)$ is torsion, for any $x_0 \in J(\overline{\mathbb{Q}}_p)$ $\exists\ n \in \mathbb{N}$ s/t $nx_0 \in \mathcal{U}$, so define $\mathcal{L}og(x_0) = \frac{1}{n}\mathcal{L}og(nx_0)$.

$\mathcal{L}og$ is locally but not globally analytic.

Now $\mathcal{L}ie\, J \cong (\Omega(X))^\vee$.

Given $\omega \in \Omega(X)$, we can form the composition

$$X \xrightarrow{i_p} J(X) \xrightarrow{\mathcal{L}og} \mathcal{L}ie\, J \xrightarrow{\omega} \mathbb{C}_p$$

We can think of this composition as

$$x \longmapsto \int_p^x \omega$$

We can define $\int_x^y \omega = \int_p^y \omega - \int_p^x \omega$

Note that this does not depend on the path.

Weird Consequence $F_\omega(x) = \int_p^x \omega$ is a global function.

This is weird. The 1-form ω should represent a non-trivial de Rham cohomology class so it shouldn't be exact. So F_ω cannot be analytic. It is only locally analytic. It has a lot of weird properties. The number of zeroes is not constrained in the same way.

There are no periods because $\mathbb{C}_p$ is not a sensitive enough field. Colmez developed an integration theory valued in Fontaine's period rings. Rephrasing of comparison theorems in p-adic Hodge theory.

Problem: You can't compute these integrals. There are number-theoretic reasons why you'd want to compute them

Why?

Let X be a curve defined over $\mathbb{Q}$.

1) Rational points. Identifying rational points on X.

This means points with rational coordinates.

Mordell-Weil Theorem Rational points on $J(X)$ are a finitely generated Abelian group.

Set $\rho(X) = \operatorname{rk} J(X)(\mathbb{Q})$.

If $p \in X(\mathbb{Q})$, $\mathcal{L}og(i_p(X(\mathbb{Q})))$ lies in a vector subspace of dimension $\leq \rho(X)$.

<u>Upshot</u> If $\rho(X) < g$, there is a 1-form ω such that $F_\omega = \int_p^x \omega$ vanishes on all rational points. Using some estimates for zeroes of F_ω in a residue disc, we can prove that for $p > g$

Then $|X(\mathbb{Q})| \leq |X_0(\mathbb{F}_p)| + (2g-2)$

We can bound $|X_0(\mathbb{F}_p)|$ by Hasse-Weil bound.

Coleman's Explicit Chabauty!

2) Torsion points

$J(X)$ is an Abelian group. Let $T \subset J(X)$ be its torsion points. The set $i_p(X) \cap T$ is called the torsion points of the curve.

Consider $\mathcal{L}og: J(X) \longrightarrow \operatorname{Lie} J$. Since $\operatorname{Lie} J$ is torsion-free, $\mathcal{L}og(T) = 0$.

<u>Upshot</u> $F_\omega(x) = \int_p^x \omega$ vanishes on torsion points.

Coleman Integrals

Coleman, inspired by Dwork, found a way to "analytically continue" integrals.

"Rigid analysis was created to provide some coherence in an otherwise totally disconnected p-adic realm. Still, it is often left to Frobenius to quell the rebellious outer provinces."

If X has good reduction, then $F: X_0 \to X_0$. After removing some discs $D_1, \ldots, D_r \subset X$ to get $U = X \setminus D_1, \ldots, D_r$, we can extend F to $F: U \to U$.

Why does this help us define integrals?

A reasonable integration theory should obey a change of variables formula. If $g: U \to U$,

$$\int_p^q g^* \omega = \int_{g(p)}^{g(q)} \omega.$$

Property of F: Given $x \in X$, $\bar{x} \in X_0$ is reduction. If $\bar{x}$ has coordinates in $\mathbb{F}_p$, then $F(x)$ is in the same residue disc as x. So $\int_x^{F(x)} \omega$ can be computed by antidifferentiating.

$$\text{So} \quad \int_p^q F^* \omega = \int_{F(p)}^{F(q)} \omega.$$

We can treat Frobenius as a map on $H^1_{dR}(U)$. Pick a basis $[\omega_1], \ldots, [\omega_r]$ for $H^1_{dR}(U)$. There's a matrix M with $\mathbb{C}_p$-coefficients representing $F^*: H^1_{dR}(U) \to H^1_{dR}(U)$. Then there are analytic functions $G_1, \ldots, G_r$ on U such that

$$F^*\Bigg(\underbrace{\begin{bmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_r \end{bmatrix}}_{= \vec{\omega}}\Bigg) = M \begin{bmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_r \end{bmatrix} + d \underbrace{\begin{bmatrix} G_1 \\ G_2 \\ \vdots \\ G_r \end{bmatrix}}_{= \vec{G}}.$$

Suppose p and $F(p)$ are in the same residue disc. Ditto for q and $F(q)$. Then,

$$\int_{F(p)}^{F(q)} \vec{\omega} = \int_p^q F^* \vec{\omega} = M \int_p^q \vec{\omega} + G(q) - G(p).$$

r-tuple of integrals

Now, $\displaystyle\int_{F(p)}^{F(q)} \vec{\omega} = \int_{F(p)}^{p} \vec{\omega} + \int_p^q \omega + \int_q^{F(q)} \omega$, so

$$(I - M)\int_p^q \vec{\omega} = \vec{G}(q) - \vec{G}(p) - \int_{F(p)}^{p} \vec{\omega} - \int^{F(q)} \vec{\omega}.$$

By Weil Conjectures, we can invert $I - M$ and solve for $\int_p^q \omega$.

This gives local description of $\int \omega$. Explicit estimates! Work of Jennifer Balakrishnan et al.

Bad Reduction Curves

X is (potential) bad reduction if there does not exist a smooth model $\mathcal{X}/\mathcal{O}_K$ for some extension K.

We can still achieve semistable (at worst nodal singularities in $X_0/\mathbb{F}_p$) after a finite extension.

Abelian integrals still work.

Coleman integrals?

If X is bad reduction, there is not a lift of Frobenius. There's still an action on $H^1_{dR}(U)$ but it has unit eigenvalues, so $I-M$ may not be invertible.

Berkovich-Coleman Integrals:

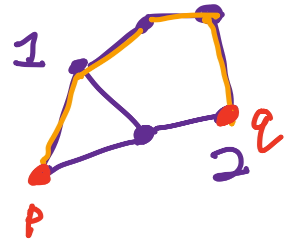
1) Cover curve X by annuli and by open sets U_i that can be embedded in good reduction curves

Corresponds to a semistable model:

2) Piece integrals together using on annuli

$$\int \frac{dz}{z} = \mathrm{Log}(z) \text{ for a branch of}$$

p-adic logarithm on K^*



3) These have periods and are not equal to Abelian integrals. We can correct them to Abelian integrals:

a) Stoll – local analysis
b) Besser–Zerbes – height pairings
c) K–Rabinoff–Zureick-Brown – Raynaud uniformization + tropical geometry.

Get univalent integrals.

Implemented as algorithm for hyperelliptic curves by K–Kaya.

Periods are Tate periods (Tate curves) Drinfel'd–Manin periods (Schottky curves)

<u>Iterated Integrals</u>

K. T. Chen: collection of holomorphic 1-forms $\omega_1, \dots, \omega_r$

$$\int \omega_1 \omega_2 \dots \omega_r$$ defined iteratively in classical situation

$$\int \omega_1 = F_{\omega_1}$$

$$F_{\omega_1\omega_2} = \int \omega_1\omega_2 = \int F_{\omega_1}\omega_2$$

$$\int \omega_1\omega_2\omega_3 = \int F_{\omega_1,\omega_2}\omega_3 \quad \text{etc.}$$

The p-adic analogue is important for Kim's non-Abelian Chabauty.

How do we define these in bad reduction case?

Besser's Tannakian reformulation in good reduction case:

1) Work in rigid setting with $X_0/\mathbb{F}_{q^r}$

(pretend we have a lift X/K and use de Rham cohomology)

2) There's a Tannakian fundamental group

$$\pi_1^{rig}(X_0; \bar{p}, \bar{q})$$

classifies "parallel transport operations" on iterate extensions of trivial bundle $\mathcal{O}$.

Think: $\int$ is parallel transporting sections of $\mathcal{O}^{\oplus r+1}$ with connection s/t we're solving

$$d\begin{bmatrix} S_0 \\ S_1 \\ S_2 \\ \vdots \\ S_r \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ \omega_1 & 0 & 0 & \cdots & 0 \\ 0 & \omega_2 & 0 & \cdots & 0 \\ & & \ddots & \ddots & \\ & 0 & & \omega_r & 0 \end{bmatrix} \begin{bmatrix} S_0 \\ S_1 \\ S_2 \\ \vdots \\ S_r \end{bmatrix}$$

3) By analysis of weights on π_1, there's a unique Frobenius invariant path, γ_{pq}

$$\int_{\gamma_{pq}} \omega_1 \cdots \omega_r \text{ is parallel transport by } \gamma_{pq}.$$

Bad Reduction Case

There is not a Frobenius invariant path.

Vologodsky: Look at $\Pi_1 = \widehat{K[\pi_1]}$

It carries monodromy operator

$$N : \Pi_1 \circlearrowleft$$

Vologodsky picks out Frobenius invariant path for which power of monodromy vanishes to a certain order.

Not explicit. Connection to Berkovich integration?

Berkovich (not in this language): there's a specialization map $sp : \pi_1(X; p, q) \longrightarrow \pi_1(\Gamma; P, Q)$ (P, Q sp of p, q)

s/t

$$sp : \pi_1(X; p, q)^{\phi} \xrightarrow{\cong} \pi_1(\Gamma; P, Q)$$

Frobenius invariants.

Given $\gamma \in \pi_1(\Gamma; P, Q)$, there's a unique Frobenius invariant lift $\gamma \in \pi_1(X; p, q)$.

This gives Berkovich-Coleman integration.

Univalent integration?

We will pick out a unique linear combination of paths $\gamma \in \pi_1(\Gamma; P, Q)$ whose lift will give Vologodsky integral. Leads to univalent integrals

<u>Combinatorial Integrals</u> (Cheng-K) based on ideas of Mikhalkin + Chen

Tropical 1-form η on Γ:

1) On edges $e \cong [0, 1]$, $\eta = a_e\, dt$
2) Harmonic: for each $v \in V(\Gamma)$

$$\sum_{e} a_e = 0$$

e, edges directed away from v

On Γ, define $\int_\gamma \eta_1 \cdots \eta_r$ as above (piecewise polynomial primitives).

There is a unique path $\gamma \in \widehat{\pi}(\Gamma; P, Q)$ for which all tropical integrals vanish. Its lift to $\pi_1^{rig}(X_0; P, Q)^{\phi}$ is the Vologodsky $\int$.

Upshots:

1) Agrees with Stoll, et al.
2) This is the unique univalent path coming from tropical geometry.

Applications to non-Abelian Chabauty? Already work of Betts - Dogra on non-Abelian Kummer map

Height pairings? Work of Besser et. al.