Brauer groups

k field

$Br(k)$ Brauer group

- central simple algebras $/ k$

  composition: tensoring

  equivalence: Morita $\qquad Mat_n(R) \sim R$ for any ring R

- cohomological interpretation: $H^2(\overset{G_k}{k}, \mathbb{G}_m)$ Galois cohom.

  for scheme S, $Br(S) = H^2_{et}(S, \mathbb{G}_m)_{tors}$ etale cohom.

- geometric interpretation:

  Brauer-Severi varieties: twisted forms of proj space

  idea $\underbrace{H^1(k, PGL_n(k_s))}_{\text{classifies}} \hookrightarrow Br(k)$

  classifies — twisted forms of $\mathbb{P}^{n-1}$

  — twisted forms of $Mat_{n \times n}$

  (SES: $1 \to \mathbb{G}_m \to GL_n \to PGL_n \to 1$)

Splitting Brauer classes

$\alpha \in Br(k)$ split by a ~~field K/k~~ scheme X means $\underset{\underset{\alpha_X}{\parallel}}{\varphi^* \alpha} = 0$ in $Br(\cancel{K}\, X)$

$$\begin{array}{ccc} \overset{X}{\cancel{\mathrm{Spec}\, K}} & \xrightarrow{\varphi} & \mathrm{Spec}\, k \\ \underset{Br(X)}{\cancel{Br(K)}} & \xleftarrow{\varphi^*} & Br(k) \end{array}$$

equivalently $\exists$ rational map $X \dashrightarrow V(\alpha) \leftarrow$ Brauer-Severi variety assoc to α

"nice"

smooth (proj geom. conn.)

$\underline{\text{def}}$ index$(\alpha) := \sqrt{|{}^{\text{div}}_{\text{alg}} : k|}$ twisted Mat$_n$ $\rightsquigarrow$ index n

period$(\alpha) :=$ order of α in $Br(k)$

period | index, same prime factors

Q/ What varieties split $\alpha \in Br(k)$? (fix k, fix $\alpha \leftrightarrow V$)

$\underline{\text{dim } 0}$ pure alg question, $\underline{\text{index}}$

$\underline{\text{dim } 1}$ $\underline{\text{genus } 0}$ "easy" if $C = \mathbb{P}^1$, then only splits $\alpha = 0$

else, only splits "itself" and 0

$\boxed{\text{genus 1}}$ $\underline{\text{hard}}$! $\boxed{Q}$ $\exists$ smooth genus one curve $C \longrightarrow V$?

* $\underline{\text{index 2}}$ anticanonical bundle L - deg 2 on V

section of $L^{\otimes 2}$ + cyclic covering trick

* $\underline{\text{index 3}}$ section of anticanonical on V

* $\underline{\text{index 4}}$ $\cap$ 2 quadrics in $\mathbb{P}^3$ is (usually) a genus one curve

[dJ-H]

$$\begin{array}{c} \mathbb{P}^3 \times \mathbb{P}^1 \\ \pi \big\downarrow \text{projection} \\ \mathbb{P}^3 \end{array}$$

zero locus of $\pi_* \mathcal{O}_{\mathbb{P}^3 \times \mathbb{P}^1}(2,1)$

α index $4 \Rightarrow$ period 2 or 4

$\Rightarrow 2\alpha$ period 1 or 2

$$\begin{array}{ccc} V & \times & Y \\ \uparrow & & \uparrow \\ \alpha & & 2\alpha \end{array}$$

2α B-S variety of dim 1

$\mathcal{O}_{V_{\bar{k}}}(2) \boxtimes \mathcal{O}_{Y_{\bar{k}}}(1)$ on $V_{\bar{k}} \times Y_{\bar{k}}$

does this descend to $V \times Y$?

obstruction: low deg exact seq

$$0 \to \mathrm{Pic}(Z) \to \mathrm{Pic}(Z_{\bar{k}})^{G_k} \to \mathrm{Br}\, k \to \mathrm{Br}\, Z \to \cdots$$

apply to $Z = V, Y$

$\mathcal{O}_{V_{\bar{k}}}(1)$: α

$\mathcal{O}_{V_{\bar{k}}}(2)$: 2α

$\mathcal{O}_{Y_{\bar{k}}}(1)$: 2α

Kunneth
$2\alpha + 2\alpha = 4\alpha = 0$
obstruction for $\mathcal{O}_{V_{\bar{k}}}(2) \boxtimes \mathcal{O}_{Y_{\bar{k}}}(1)$

* index 5 use Buchsbaum-Eisenbud/Fisher
* index 6 (Auel)
* general index? OPEN

higher genus curves

index n B-S var V ← dim $n-1$

complete intersections of anticanonical bundle sections (deg n)

if smooth

$$g = 1 + \frac{1}{2} n^{n-1}(n-3)$$

Q better bounds?

torsors under abelian varieties (jt w/ M. Lieblich)

Thm k field, $\alpha \in \mathrm{Br}(k)$

$\exists$ torsor T under AV over k s.t. T splits α

$\Uparrow$

Almost thm If $\alpha_C = 0$ for some nice curve C of genus ≥ 1, then Alb_C splits α.

$\parallel$

$\mathrm{Pic}^1_{C/k}$

OK if $\operatorname{index}(\alpha) \not\equiv 2 \bmod 4$

if $\operatorname{ind}(\alpha) \equiv 2 \bmod 4$, add a C' factor (genus 1)

$C' \times \mathrm{Alb}_C$ splits α.

some reductions · wlog $g = g(C) \geq 2$

· $\deg K_C = 2g-2$ and $\alpha_C = 0 \Rightarrow \operatorname{index}(\alpha) \mid 2g-2 = 2(g-1)$

$\Rightarrow$ if $\operatorname{ind}\alpha \not\equiv 2\ (4)$, then $(g, \operatorname{ind}(\alpha)) = 1$.

Sketch of pf

$$0 \to \mathrm{Pic}\, Z \to (\mathrm{Pic}\, Z_{\bar{k}})^{G_k} \xrightarrow{obs} \mathrm{Br}\, k \to \mathrm{Br}\, Z$$

compare kernels of obstr for C and $\mathrm{Alb}_C =: X$

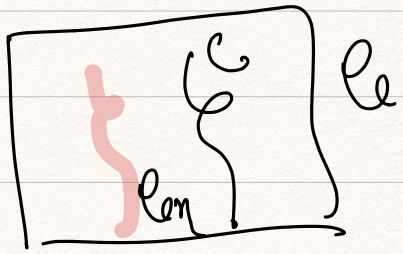
lemma C nice curve/k $g \geq 2$

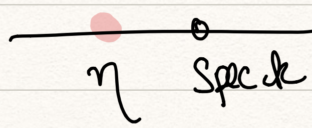
$X = \mathrm{Alb}_C$

Assume $\alpha \in \mathrm{Br}\, k$ splits by C and has period prime to g.

If $NS(X_{\bar{k}}) \hookrightarrow NS(C_{\bar{k}}) \cong \mathbb{Z}$ w/ image $\supseteq g \cdot NS(C_{\bar{k}})$ then X splits α.

lemma the generic curve of any $g \geq 2$ has this NS property





B regular local ring of M_g

$$\begin{tikzcd}
C \text{ splits } \alpha \arrow[r, Rightarrow] \arrow[d, Rightarrow, "\text{def thm}"] & \mathrm{Alb}_C \text{ splits } \alpha \\
\mathcal{C} \text{ splits } \alpha \arrow[d, Rightarrow, "*"] & \mathrm{Alb}_{\mathcal{C}} \arrow[u, Rightarrow, "*"'] \\
\mathcal{C}_\eta \arrow[r, Rightarrow, "\text{lemma}"] & \mathrm{Alb}_{\mathcal{C}_\eta} \arrow[u, Rightarrow, "\text{regularity}"']
\end{tikzcd}$$

Q/ better dim bounds?