

Wall-Crossing for Newton-Okounkov bodies

joint work with Megumi Harada

arXiv: 1912.04809

Outline I. Polytopes & algebraic geometry

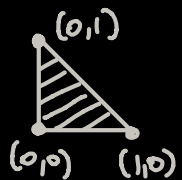
II. Newton-Okounkov bodies (NO-bodies)

III. Wall-crossing for NO-bodies

IV. The tropical Grassmannian $\text{Gr}(2, m)$

The Newton polytope of $f = \sum_a c_a x^a \in \mathbb{C}[x_1, \dots, x_n]$ is

$$\text{Newt}(f) := \text{conv} \{ a \mid c_a \neq 0 \}.$$

$$\text{Newt}(3x + y - 1) = \triangle$$


Bernstein-Khovanskii-Kuchnirenko Thm:

$$A = \{ a_1, \dots, a_s \} \subseteq \mathbb{Z}^n$$

$$L_A = \left\{ \sum_{i=1}^s c_i x^{a_i} \mid c_i \in \mathbb{C} \right\}$$

For a generic choice of $f_1, \dots, f_n \in L_A$, the number of solutions to $f_1(x) = \dots = f_n(x) = 0$ in $(\mathbb{C}^*)^n$ is equal to $n! \text{vol}(\text{conv}(A))$.

II. NO-bodies

Δ a polytope $\rightsquigarrow X_\Delta \subseteq \mathbb{P}^s$ a projective toric variety

Thm: $\deg(X_\Delta) = n! \cdot \text{vol}(\Delta)$

NO-bodies: input: $X \subseteq \mathbb{P}^s$ a projective variety
a valuation on $\mathbb{C}[X]$

output: a convex body Δ st $\text{vol}(\Delta) = \deg(X)$

[Okounkov, Lazarsfeld-Mustaŭă, Kahen-Khovanskii]

Valuations

Equip $\mathbb{Z}^n$ with a total order $\geq$

$\nu: \mathbb{C}[X] \setminus \{0\} \rightarrow \mathbb{Z}^n$ is a valuation if

- i) $\nu(f+g) \geq \max\{\nu(f), \nu(g)\}$
- ii) $\nu(fg) = \nu(f) + \nu(g)$
- iii) $\nu(c) = 0$ for all $c \in \mathbb{C} \setminus \{0\}$

Example: $\nu\left(\sum_a c_a X^a\right) := \min\{a \mid c_a \neq 0\}$ is a valuation on $\mathbb{C}[x_1, \dots, x_n]$

No-body $\Delta(X, \nu) := \overline{\text{cone}(\text{image}(\nu))} \cap \{x_i = 1\}$

Thm [Okounkov, Lazarsfeld-Mustaŭa, Kaveh-Khovanskij]:

$$\deg(X) = n! \text{vol}(\Delta(X, \nu)).$$

An example

X the hypersurface given by $I = \langle y^2z - x^3 + 7xz^2 - 2z^3 \rangle$.

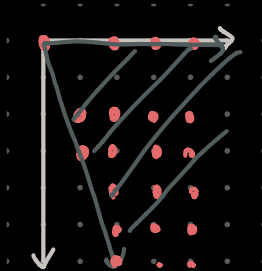
let ν be the following valuation:

$$M = \begin{bmatrix} 1 & 1 & 1 \\ -2 & -3 & 0 \end{bmatrix} \rightsquigarrow \nu(x^\alpha y^\beta z^\gamma) = M \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$$

$$\nu\left(\sum c_{\alpha\beta\gamma} x^\alpha y^\beta z^\gamma\right) = \min\{\nu(x^\alpha y^\beta z^\gamma) \mid c_{\alpha\beta\gamma} \neq 0\}$$

image(ν) = semigroup generated
by the columns of M

$$\Delta(X, \nu) = \overline{\text{core}(\text{image}(\nu))} \cap \{x_i = 1\}$$



An example

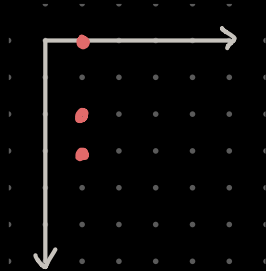
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image(ν) = semigroup generated
by the columns of M



$\Delta(X, \nu)$ = convex hull of the columns of M

Anderson: When $\Delta(X, \nu)$ is a rational polytope, there is a degeneration of X to the toric variety of $\Delta(X, \nu)$.
i.e. a one-parameter flat family $\pi: \mathcal{X} \longrightarrow \mathbb{A}^1$
with $\pi^{-1}(1) = X$ and $\pi^{-1}(0) = X_{\Delta(X, \nu)}$.

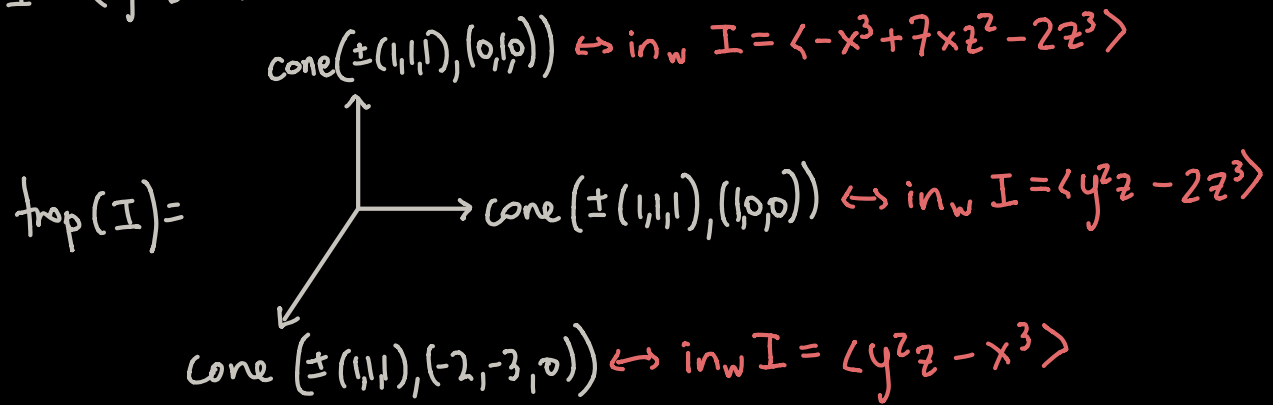
Kaveh-Manon: construct valuations with $\Delta(X, \nu)$ a rational polytope using tropical geometry.

The tropicalization of I is $\text{trop}(I) := \{w \in \mathbb{Q}^n \mid \text{in}_w I \text{ contains no monomial}\}$

$\text{trop}(I)$ has the structure of a fan:

the open cones are $C_w = \{w' \in \text{trop}(I) \mid \text{in}_{w'} I = \text{in}_w I\}$.

Example: $I = \langle y^2z - x^3 + 7xz^2 - 2z^3 \rangle$



A cone C in $\text{trop}(I)$ is prime if $\text{in}_w I$ is prime for some/any $w \in C^\circ$.

Let C be a prime cone of $\text{trop}(I)$ and $\{u_1, \dots, u_r\} \subseteq C$ be maximally linearly independent. **Kaveh-Manon** construct a valuation $\nu: \mathbb{C}[x_1, \dots, x_n]/I \longrightarrow \mathbb{Z}^r$ whose No-body is a rational polytope

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Explicitly, $\Delta(V(I), \nu)$ is the convex hull of the columns of the matrix $\begin{bmatrix} -u_1 & - \\ \vdots & \\ -u_r & - \end{bmatrix}$

Example: $I = \langle y^2z - x^3 + 7xz^2 - 2z^3 \rangle$
 $C = \text{cone}(\pm(1, 1, 1), (-2, -3, 0))$.

$$M = \begin{bmatrix} 1 & 1 & 1 \\ -2 & -3 & 0 \end{bmatrix}$$

$$\Delta(X, \nu) = \text{conv}\{(1, -2), (1, -3), (1, 0)\}$$

Geometric Wall-Crossing for ND-bodies

Thm [E.-Harada]: let C_1, C_2 be two prime cones of $\text{trop}(I)$ s.t.

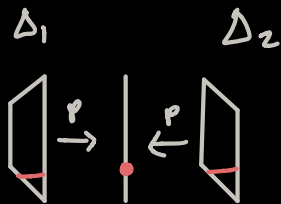
① C_1, C_2 are of maximal dimension

② C_1, C_2 share a codim 1 face

let $\{u_{i1}, \dots, u_{id}\} \subseteq C_i$ be maximally linearly ind. and such that
 $u_{11} = u_{21}, \dots, u_{1d-1} = u_{2d-1}, u_{1d} \neq u_{2d}$.

Then ① the fibers of the projection $p: \mathbb{R}^d \rightarrow \mathbb{R}^{d-1}$

restricted to $\Delta(X, \nu_1)$ & $\Delta(X, \nu_2)$ are
intervals of the same length.



Geometric Wall-Crossing for NO-bodies

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Then ② We obtain 2 maps $\Delta(x, \nu_1) \rightarrow \Delta(x, \nu_2)$

one that shifts the interval

& another that vertically reflects $\Delta(x, \nu_1)$ & then shifts.

Remarks:

- ① Ilten-Manon observed in 2017 that this geometric wall-crossing can be derived from the theory of complexity-one T -varieties
- ② Ilten interprets geometric wall-crossing as a generalization of the "combinatorial mutation" of Akhtar-Coates
- Galkin-Kaprzyk.
- ③ E.-Harada give an "algebraic wall-crossing", i.e. a map between the semigroups $\text{image}(\nu_1), \text{image}(\nu_2)$.

Idea of proof:

① Since C_i is prime & max'l dimensional $\Rightarrow \text{inc}_i I$ is toric,
& the variety for this ideal is the toric variety of $\Delta(x, v_i)$.

Let $X_i = \text{TV of } \Delta(x, v_i)$ & $Y = \text{variety of } \text{inc}_1, \text{inc}_2 I$.

② There exist flat families $\mathcal{X}_1, \mathcal{X}_2$ over $\mathbb{A}^1$ w/ generic fibers isom to Y & special fibers isomorphic to X_1, X_2 (respectively).

③ There is a codim-1 torus $T \hookrightarrow Y$

This torus is a subtorus of the torus $T_i \hookrightarrow X_i$

The TV of a fiber $p^{-1}(\xi) \in \Delta(x, v_i)$ is the GIT quotient of X_i by T at ξ .

④ T-action extends to the families $\mathcal{X}_1, \mathcal{X}_2$

By taking GIT quotient of $\mathcal{X}_i$ by T at ξ we obtain

2 flat families w/ generic fiber $Y //_{\xi} T$

special fibers $X_i //_{\xi} T$ (resp.)

⑤ flat families preserve degree

$$\Rightarrow \deg(X_1 //_{\xi} T) = \deg(Y //_{\xi} T) = \deg(X_2 //_{\xi} T)$$

$\parallel$ $\parallel$

length of fiber length of fiber

of ξ in of ξ in

$\Delta(X, \nu_1)$ $\Delta(X, \nu_2)$

Case Study: The tropical Grassmannian $\text{Gr}(2, m)$

$\text{Gr}(2, m)$ = space of 2-planes in $\mathbb{C}^m$

$I_{2, m}$ = plücker ideal of $\text{Gr}(2, m)$

$$I_{2, 4} = \langle P_{12}P_{34} - P_{13}P_{24} + P_{14}P_{23} \rangle$$

☆ All maximal cones of $\text{trop}(I_{2, 4})$ are prime.

Remark: $\mathbb{C}[\text{Gr}(2, m)]$ is a cluster algebra

Roughly: $\text{Gr}(2, m)$ has an atlas of tori

transition maps between the tori are cluster mutations

Our flip wall-crossing is a tropicalized toric mutation.

Thank you !

¡Muchas gracias !